

Quantum BRST charge in gauge theories in curved space-time

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Abstract

Renormalized gauge-invariant observables in gauge theories form an algebra which is obtained as the cohomology of the derivation $[\mathbf{Q}_L, -]$ with \mathbf{Q}_L the renormalized interacting quantum BRST charge. This quotient algebra then admits a Hilbert space representation. For a large class of gauge theories in Lorentzian globally hyperbolic space-times, we derive an identity in renormalized perturbation theory which expresses the commutator $[\mathbf{Q}_L, -]$ in terms of a new nilpotent quantum BRST differential and a new quantum anti-bracket which differ from their classical counterparts by certain quantum corrections. This identity enables us to prove different manifestations of gauge symmetry preservation at the quantum level in a model-independent fashion.

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1 Introduction

Quantum field theories with local gauge symmetry play a crucial role in our understanding of elementary particle physics by describing the type of interactions between them. The quantum aspects of such theories in flat space-time have been extensively studied. To describe the elementary particles in the Early Universe where the curvature of space-time is not negligible, one needs to extend the framework of flat space gauge theories to the curved space setting. In [1], it was shown that the renormalized quantum Yang-Mills theories in an arbitrary, Lorentzian, globally hyperbolic curved space-time can be consistently constructed to all orders in perturbation theory. However, the proof of the statements in that reference rests on the specific form of the pure Yang-Mills interaction. The present work aims to investigate in a more model-independent fashion the issue of symmetry preservation at the quantum level and the gauge-fixing independence of renormalized quantum gauge theories in curved space-times. For concreteness, we work with the pure Yang-Mills theory, however our results rest only on a certain aspect of this theory, namely the absence of “gauge anomaly”, which is also the case in a larger class of more complicated theories with local gauge symmetry. For instance, our results remain valid for superconformal Chern-Simons matter theory in 3 dimensions [2], a class of superconformal gauge theories in 4 dimensions [3], and perturbative quantum gravity [4], [5].

Contrary to the flat space-time setting where the quantum fields can be represented as operators on a preferred Hilbert space containing the unique Poincaré-invariant vacuum state, in a generic (globally hyperbolic) curved space-time there is no preferred vacuum state and hence no canonical Hilbert space representation of the theory. We therefore employ the framework of locally covariant quantum field theory [6][7],[8], [9], [10] (see [11] for a recent review) to study such theories. In this framework, one formulates the QFT coherently on all space-times and views the renormalized composite quantum fields \mathcal{O}_L , under interaction L , as elements of an abstract algebra, which can be constructed in perturbation theory, and may be represented on a (non-canonical choice of a) Hilbert space at the end. (This choice of representation is related to the physical questions one wants to study and is not discussed here.)

In theories with local gauge invariance, the field equations are not of hyperbolic type and hence a straightforward perturbative quantization is not possible. An elegant way out of this problem is the BRST method [12] [13]. In this approach, one constructs a “gauge-fixed” and enlarged action which yields hyperbolic field equations, and exhibits the nilpotent BRST symmetry \hat{s} . At the quantum level, local gauge symmetry has the following different manifestations:

- (1) conservation of the renormalized interacting Noether current \mathbf{J}_L of BRST symmetry,
- (2) nilpotency of $[\mathbf{Q}_L, -]$ generated by BRST charge \mathbf{Q}_L (obtained from \mathbf{J}_L),
- (3) invariance of the “local S-matrix” \mathcal{S} under this derivation: $[\mathbf{Q}_L, \mathcal{S}] = 0$,
- (4) invariance of renormalized operators $[\mathbf{Q}_L, \mathcal{O}_L] = 0$, for gauge-invariant \mathcal{O} ,

Let us elaborate on these points: (1) is a prerequisite for (2). By (2) we can observe that the quantization of the enlarged theory gives an algebra of interacting fields which can be represented only on a space with indefinite inner product. One then defines the physical algebra of gauge-invariant observables, which admits a Hilbert space representation, as the cohomology of $[\mathbf{Q}_L, -]$. In our local approach, we work with a local, infrared cutoff

S-matrix which may lead to an scattering matrix via an “adiabatic limit” (if exists) and then (3) ensures the gauge invariance of the scattering amplitudes in a Hilbert space representation. (4) expresses the natural requirements that the renormalized gauge-invariant observables in the classical theory must remain gauge-invariant in the quantum theory

Our main result in the present work is a universal, case-independent proof that indeed all the above manifestations follow from the fulfilment of a single renormalization condition formulated in [1] and called the “Ward identity”. The possible violation of this identity corresponds to a certain cohomology class of \hat{s} containing “gauge anomalies”. In fact, in [1] requirements (2) and (4) are shown to hold based on a case-dependent and complicated proof which requires, in addition to the Ward identity, the precise form of the current of pure Yang-Mills theory and certain identities derived from it, as well as the triviality of a higher cohomology class which seem to hold only for this specific theory.

The key identity that we derive in this work, and which forms the basis of our proofs of claims (2) - (4), expresses the commutator of the quantum interacting BRST charge \mathbf{Q}_L with expressions of the form $T_{L,n}(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n)$. Here, $(T_{L,n})_{n \geq 1}$ are the set of renormalized interacting time-ordered products, and $\mathcal{O}_1, \dots, \mathcal{O}_n$ are arbitrary local operators, with $T_{L,1}(\mathcal{O}) = \mathcal{O}_L$. When evaluated in a state, such expressions give the renormalized time-ordered correlation functions of the theory. Explicit calculations for one and two local operators, as proved in corollary 19, give

$$[\mathbf{Q}_L, \mathcal{O}_L(x)] = i\hbar(\hat{q}\mathcal{O}(x))_L, \quad (1)$$

$$[\mathbf{Q}_L, T_{L,2}(\mathcal{O}_1(x) \otimes \mathcal{O}_2(y))] = i\hbar T_{L,2}(\hat{q}\mathcal{O}_1(x) \otimes \mathcal{O}_2(y) + \mathcal{O}_1(x) \otimes \hat{q}\mathcal{O}_2(y)) + \hbar^2((\mathcal{O}_1(x), \mathcal{O}_2(y))_{\hbar})_L. \quad (2)$$

The new algebraic structures which arise in our formulation are the *quantum BRST differential*

$$\hat{q}\mathcal{O} := \hat{s}\mathcal{O} + \hat{A}_1(\mathcal{O}), \quad (3)$$

which is nilpotent, i.e. $\hat{q}^2 = 0$, and the *quantum anti-bracket*

$$(\mathcal{O}_1, \mathcal{O}_2)_{\hbar} := (\mathcal{O}_1, \mathcal{O}_2) + \hat{A}_2(\mathcal{O}_1 \otimes \mathcal{O}_2), \quad (4)$$

which is compatible with \hat{q} , i.e.

$$\hat{q}(\mathcal{O}_1, \mathcal{O}_2)_{\hbar} = (\hat{q}\mathcal{O}_1, \mathcal{O}_2)_{\hbar} - (\mathcal{O}_1, \hat{q}\mathcal{O}_2)_{\hbar}. \quad (5)$$

In the above expressions, \hat{A}_1 , and \hat{A}_2 are two elements of a hierarchy of operators $(\hat{A}_n)_{n \geq 1}$, which are of order $O(\hbar)$ and correspond to the anomaly with n insertions.

Notations

In the body of the paper, we encounter local operators $\mathcal{O} = \mathcal{O}_0 + \lambda\mathcal{O}_1 + \lambda^2\mathcal{O}_2 + \dots$ which are p -forms expanded into powers of the coupling constant λ . For the integrated operators we use $F = \int \mathcal{O}_0 + f\lambda\mathcal{O}_1 + f^2\lambda^2\mathcal{O}_2 + \dots$ where $f \in \Omega_0^{4-p}(M)$ is an IR cutoff which is equal 1 in some region of interest. We symbolically write all such expressions as $F = \int f\mathcal{O}$. For the particular case of \mathbf{L}_{int} , the interaction Lagrangian, we denote the “cutoff interaction” with $L = \int f\mathbf{L}_{\text{int}}$, and the “true interaction” with $I = \int \mathbf{L}_{\text{int}}$. We always write the interacting BRST charge with the cutoff interaction \mathbf{Q}_L , and avoid using \mathbf{Q}_I (which

can be defined as the algebraic adiabatic limit of \mathbf{Q}_L at the end of calculations). Moreover, in many places, we write $\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n$ as a short form for $\mathcal{O}_1(x_1) \otimes \cdots \otimes \mathcal{O}_n(x_n)$. Finally, in different expressions, we encounter objects with even or odd Grassmann parity which come with a certain $+$ or $-$ sign when they are acted on by graded derivations. However, in most of such expressions the correct sign does not play any important role in making our main conclusions. To avoid clutter, we take all the operators \mathcal{O}_i in the body of the work to be bosonic (Grassmann even), and explain in appendix A the correct signs for those formulas which play a role in the proof of the main results.

2 Classical gauge theory

We take for definiteness the example of the pure Yang-Mills theory on a globally hyperbolic space-time (M, g) worked out in [1]. It turns out that, the results of this work can be generalized to all theories with local gauge symmetry, such as superconformal Yang-Mills theory [3], and superconformal Chern-Simons-matter theory [2] in which a certain cohomology class is trivial. Here, we briefly review the setting for the classical theory.

The classical Yang-Mills theory with gauge group G is the dynamical theory of a G -gauge connection $\mathcal{D} = \nabla + i\lambda A$, where ∇ is the Levi-Civita connection on (M, g) and A is a \mathfrak{g} -valued one form. The action functional is given by

$$S_{\text{YM}} = -\frac{1}{2} \int_M \text{tr}(F \wedge *F), \quad (6)$$

where F is the curvature of \mathcal{D} . For the purpose of perturbative quantization, the equations of motion for A generated by the action have to be of hyperbolic type. However, this is not the case for S_{YM} ; one has to fix the gauge in order to render the free field equations hyperbolic. The resulting gauge-fixed theory enjoys the BRST symmetry \hat{s} , if one augments the field content of the theory by further dynamical fields (ghosts) and non-dynamical fields (anti-fields), as introduced below.

Let $\{T_I\}$, $I = 1, 2, \dots, \dim \mathfrak{g}$ be a basis for the Lie algebra \mathfrak{g} of the Lie group G . Relative to this basis, we have $A = A^I T_I = A_\mu^I T_I dx^\mu$. Let us denote the set of all dynamical fields by $\Phi = (A^I, C^I, \bar{C}^I, B^I)$, where C, \bar{C} are called ghosts and B is an auxiliary field with algebraic equations of motion, and their corresponding anti-fields by $\Phi^\dagger = (A_I^\dagger, C_I^\dagger, \bar{C}_I^\dagger, B_I^\dagger)$. The action of the BRST differential \hat{s} on all fields is given by:

$$\hat{s}A_\mu^I = D_\mu C^I, \quad \hat{s}C^I = -\frac{i\lambda}{2} f^I_{JK} C^J C^K, \quad \hat{s}\bar{C}^I = B^I, \quad \hat{s}B^I = 0. \quad (7)$$

One can now assign a “ghost number” to all the above fields which is given in table 1. The ghost number defines a grading on the space of all fields, and the BRST differential increases the ghost number by one unit while leaves the dimension unchanged. To define how \hat{s} acts on the anti-fields, consider the following extended action

$$\hat{S} = S_{\text{YM}} + \hat{s}\psi - \int \hat{s}\Phi \wedge \Phi^\dagger, \quad (8)$$

where ψ is the “gauge-fixing fermion”. It is chosen in such a way that \hat{S} gives rise to hyperbolic field equations for all fields Φ . A conventional choice for ψ is $\psi = \int_M \bar{C}_I (\nabla^\mu A_\mu^I + \frac{1}{2} B^I)$ which implements the Feynman gauge. Now, for any observable \mathcal{O} we define

$$\hat{s}\mathcal{O} = (\hat{S}, \mathcal{O}), \quad (9)$$

Fields	A^I	C^I	\bar{C}^I	B^I	A_I^\dagger	C_I^\dagger	\bar{C}_I^\dagger	B_I^\dagger
Dimension	1	0	2	2	3	4	2	2
Ghost number	0	1	-1	0	-1	-2	0	-1
Grassmann parity	0	1	1	0	1	0	0	1

Table 1: Basic fields and their data.

where $(-, -)$ is the so-called anti-bracket defined by

$$(\mathcal{O}_1, \mathcal{O}_2) := \int_M \frac{\delta \mathcal{O}_1}{\delta \Phi(x)} \frac{\delta \mathcal{O}_2}{\delta \Phi^\dagger(x)} - \frac{\delta \mathcal{O}_1}{\delta \Phi^\dagger(x)} \frac{\delta \mathcal{O}_2}{\delta \Phi(x)}, \quad (10)$$

and satisfies the Jacobi identity $((\mathcal{O}_1, \mathcal{O}_2), \mathcal{O}_3) + \text{permutations} = 0$ (see appendix A for the correct signs when a Grassmann-valued field is present in the above relations). Note that in particular, it follows that $\hat{s}\Phi^\dagger(x) = \delta\hat{S}/\delta\Phi(x)$ and $(\Phi_i(x), \Phi^{\dagger j}(y)) = \delta_i^j \delta(x, y)$. Moreover, from the definition of \hat{S} we have

$$\hat{s}\hat{S} = (\hat{S}, \hat{S}) = \hat{s}^2 = 0. \quad (11)$$

Local and covariant fields

Definition 1. Let \mathcal{C} be the space of enlarged field configurations (Φ, Φ^\dagger) together with the metric g .

1. Let $f : M' \rightarrow M$ be an isometric embedding which preserves the causal structures. A **local-covariant functional** \mathcal{O} on \mathcal{C} satisfies

$$f^* \mathcal{O}[g, \Phi, \Phi^\dagger] = \mathcal{O}[f^* g, f^* \Phi, f^* \Phi^\dagger]. \quad (12)$$

2. $\mathbf{P}(M) = \oplus_{p,q} \mathbf{P}_q^p(M)$, where each $\mathbf{P}_q^p(M)$ is defined to be the space of all $\wedge^p(TM)$ -valued polynomial, local and covariant functionals with ghost number q .

There are two important theorems regarding the nature of $\mathbf{P}_q^p(M)$. First, the *Thomas replacement theorem* [14], which states that the dependence of every element $\mathcal{O} \in \mathbf{P}_q^p(M)$ on the metric and, at each point $x \in M$, on $\Phi(x), \Phi^\dagger(x)$ is of the form

$$\mathcal{O} = \mathcal{O}(g_{\mu\nu}(x), R^\mu{}_{\nu\rho\sigma}(x), \dots, \nabla_{(\mu_1} \dots \nabla_{\mu_k)} R^\mu{}_{\nu\rho\sigma}|_x, \nabla_{(\mu_1} \dots \nabla_{\mu_k)} \Phi|_x, \nabla_{(\mu_1} \dots \nabla_{\mu_k)} \Phi^\dagger|_x) \quad (13)$$

where $R^\mu{}_{\nu\rho\sigma}$ is the Riemann tensor. Therefore, if we assign dimension 1 to ∇_μ , we can assign a dimension to all elements of $\mathbf{P}_q^p(M)$. Second, the *algebraic Poincare lemma* [15] states that if for some $\mathcal{O} \in \mathbf{P}_q^p(M)$, $d\mathcal{O} = 0$, then there exists another $\mathcal{O}' \in \mathbf{P}_q^{p-1}(M)$ such that $\mathcal{O} = d\mathcal{O}'$. Note that this is a property of d -cohomology for functionals of Φ, Φ^\dagger , and holds even for space-times with non-trivial de Rham cohomology.

The q -th cohomology ring of \hat{s} at form degree p is defined by

$$H_q^p(\hat{s}, M) := \frac{\{\ker \hat{s} : \mathbf{P}_q^p(M) \rightarrow \mathbf{P}_{q+1}^p(M)\}}{\{\text{im } \hat{s} : \mathbf{P}_{q-1}^p(M) \rightarrow \mathbf{P}_q^p(M)\}}. \quad (14)$$

We will show in section 3.3, that the anomaly $A = \int_M a(x)$ is a formal power series in \hbar whose leading order contribution A^m is an element of $H_1^4(\hat{s}, M)$. Equivalently, the local function $a^m(x)$ belongs to the *cohomology rings of \hat{s} modulo d* defined by

$$H_q^p(\hat{s}|d, M) := \frac{\{\mathcal{O}_q^p | \hat{s}\mathcal{O}_q^p = d\mathcal{O}_{q+1}^{p-1}\}}{\{\mathcal{O}_q^p | \mathcal{O}_q^p = \hat{s}\mathcal{O}_{q-1}^p + d\mathcal{O}_q^{p-1}\}}. \quad (15)$$

Gauge-invariant observables and the Noether current

The local and covariant functionals introduced above, of course contain all possible gauge-variant functionals of the enlarged (un-physical) theory. It turns out [16] that one can recover the gauge-invariant observables of the original, physical theory as the following cohomology

$$\{\text{classical gauge-invariant observables}\} = H_0(\hat{s}, M). \quad (16)$$

According to the Noether's theorem, the invariance of \hat{S} under \hat{s} results in the existence of a current $\mathbf{J}(x) \in \mathbf{P}_1^3(M)$ (the BRST current) which is conserved $d\mathbf{J} = 0$ once the equations of motion hold. Indeed, we have

$$d\mathbf{J}(x) = (\hat{S}, \Phi^\dagger(x))(\Phi(x), \hat{S}). \quad (17)$$

Let us assume that (M, g) contains a compact Cauchy surface Σ . Then, there exists a corresponding BRST charge \mathbf{Q} , defined by

$$\mathbf{Q} = \int_M \gamma \wedge \mathbf{J}(x), \quad (18)$$

where γ is a closed 1-form on M with compact support such that $\int_M \gamma \wedge \alpha = \int_\Sigma \alpha$ for any closed 3-form α . \mathbf{Q} is indeed the generator of the BRST symmetry at classical level in the sense that

$$\hat{s} = \{\mathbf{Q}, -\}, \quad (19)$$

where $\{-, -\}$ is the (graded) Peierls bracket [17], [18] of the classical theory which is compatible with \hat{s} in the sense that

$$\hat{s}\{\mathcal{O}_1, \mathcal{O}_2\} = \{\hat{s}\mathcal{O}_1, \mathcal{O}_2\} + \{\mathcal{O}_1, \hat{s}\mathcal{O}_2\}. \quad (20)$$

3 Quantum gauge theory in curved space-time

We now turn to the quantization of the classical field theory introduced in the previous part. To this end, we employ the ideas of causal perturbation theory [19] adopted to the framework of locally covariant field theory [20], [8], [9] which aims to construct the algebra of observables of the theory. However, in our case we begin with constructing $\hat{\mathbf{W}}_I$ which is the quantization of the enlarged theory including gauge-variant and non-observable elements as a perturbation in λ around the free theory $\hat{\mathbf{W}}_0$. At the end, we will recover the algebra \mathcal{F}_I of physical gauge-invariant observables of the theory as a certain quotient, which admits a Hilbert space representation.

3.1 Free quantum theory and renormalization schemes

Let us begin with reviewing the construction of the algebra of free quantum fields $\hat{\mathbf{W}}_0$ corresponding to the enlarged theory defined by \hat{S}_0 . Here we split the extended action

$$\hat{S} = \hat{S}_0 + \lambda \hat{S}_1 + \lambda^2 \hat{S}_2 \equiv \hat{S}_0 + I, \quad (21)$$

where \hat{S}_0 is quadratic in all dynamical fields $\Phi^i = (A_\mu^I, C^I, \bar{C}^I, B^I)$, but has arbitrary dependence on all non-dynamical fields Φ_{bg} including anti-fields Φ_i^\dagger , the metric g and curvature tensors, and where $I = \int_M \mathbf{L}_{\text{int}}$ with $\mathbf{L}_{\text{int}} = \lambda \mathbf{L}_1 + \lambda^2 \mathbf{L}_2 \in \mathbf{P}_0^4(M)$. Let us also denote $S_0 = \hat{S}_0|_{\Phi^\dagger=0} = (S_{\text{YM}} + \hat{s}\psi)_0$

Consider now the free differential operator P_{ij}^0 , with $P_{ij}^0 \Phi^j(x) = \delta S_0 / \delta \Phi^i(x)$, and

$$P_{ij}^0 = \begin{pmatrix} g^{\mu\nu} \square + R^{\mu\nu} & 0 & 0 & \nabla^\mu \\ 0 & 0 & \square & 0 \\ 0 & -\square & 0 & 0 \\ \nabla^\mu & 0 & 0 & -1 \end{pmatrix}. \quad (22)$$

An important ingredient in constructing $\hat{\mathbf{W}}_0$ is an arbitrary but fixed 2-point function of Hadamard type $\omega^{ij}(x, y)$. It is a distribution on $M \times M$, which satisfies

- (1) $(P_{ik}^0 \otimes \mathbf{1}) \omega^{kj}(x, y) = 0 = (\mathbf{1} \otimes P_{ik}^0) \omega^{kj}(x, y)$,
- (2) $\omega^{ij}(x, y) - \omega^{ij}(y, x) = i \Delta^{ij}(x, y)$, where $\Delta^{ij}(x, y)$ is the causal propagator of P_{ij}^0 ,
- (3) a specific wave-front set bound (see [21]).

Explicitly,

$$\omega^{ij}(x, y) = k_{IJ} \otimes \begin{pmatrix} \omega^{\mu\nu}(x, y) & 0 & 0 & -i \nabla_\nu \omega^{\mu\nu}(x, y) \\ 0 & 0 & i \omega(x, y) & 0 \\ 0 & -i \omega(x, y) & 0 & 0 \\ -i \nabla_\mu \omega^{\mu\nu}(x, y) & 0 & 0 & 0 \end{pmatrix}, \quad (23)$$

where $\omega(x, y)$, $\omega^{\mu\nu}(x, y)$ are scalar and vector two point functions respectively (see e.g. [22]). They satisfy the following consistency relations

$$\nabla_\mu \omega^{\mu\nu}(x, y) = -\nabla^\nu \omega(x, y), \quad \nabla_\nu \omega^{\mu\nu}(x, y) = -\nabla^\mu \omega(x, y). \quad (24)$$

Definition 2 ([8], [1]).

(1) The off-shell free algebra $\hat{\mathbf{W}}_0(M, g)$ is the $*$ -algebra generated by the identity $\mathbf{1}$ and elements

$$F(u) = \int u_{k_1 \dots k_n}^{i_1 \dots i_m}(x_1, \dots, x_n; y_1, \dots, y_m) : \Phi_{i_1}(x_1) \dots \Phi_{i_m}(x_n) :_\omega \Phi_{b_g}^{k_1}(y_1) \dots \Phi_{b_g}^{k_n}(y_m).$$

In this expression,

$$\begin{aligned} & : \Phi(x_1) \dots \Phi(x_n) :_\omega \\ &= \frac{\delta^n}{i^n \delta f(x_1) \dots \delta f(x_n)} \exp_\star \left(i \int_M f(x) \Phi(x) + \frac{\hbar}{2} \int_{M^2} \omega(x, y) f(x) f(y) \right) \Big|_{f=0}, \end{aligned} \quad (25)$$

with the star product of two basic fields being defined by

$$\Phi^i(x) \star \Phi^j(y) = \Phi^i(x) \cdot \Phi^j(y) + \hbar \omega^{ij}(x, y), \quad (26)$$

and u is a distribution subject to the following wave front set condition in the variables x_1, \dots, x_n

$$WF(u) \cap \bigcup_{x \in M} [(\bar{V}_x^+)^{\times n} \cup (\bar{V}_x^-)^{\times n}] = \emptyset, \quad (27)$$

where \bar{V}_x^\pm is the closure of the future/past light cone at $x \in M$, but u is not subject to any wave front set condition in the variables y_1, \dots, y_m . The $*$ -operation, denoted by \dagger , is defined by $F(f)^\dagger = F(\bar{f})$.

(2) The on-shell free algebra $\hat{\mathcal{F}}_0(M, g)$ is the quotient

$$\hat{\mathcal{F}}_0 = \hat{\mathbf{W}}_0 / \mathcal{J}_0, \quad (28)$$

where \mathcal{J}_0 is the \star -ideal generated by the equations of motion, that is, the space of generators $F(u)$ containing a factor $\delta \hat{S}_0 / \delta \Phi$ of free equations of motion for dynamical fields.

Note that (26) implies

$$[\Phi^i(x), \Phi^j(y)] = i\hbar \Delta^{ij}(x, y) \mathbf{1}, \quad (29)$$

$$[\Phi_{\text{bg}}^i(x), \Phi^j(y)] = 0 = [\Phi^i(x), \Phi_{\text{bg}}^j(y)], \quad (30)$$

where $[\Phi^i(x), \Phi^j(y)] = \Phi^i(x) \star \Phi^j(y) \pm \Phi^j(y) \star \Phi^i(x)$ is the graded commutator. Moreover in $\hat{\mathcal{F}}_0$, the basic dynamical fields satisfy the free equations of motion $P_{ij}^0 \Phi^j(x) = 0$.

Renormalization schemes and finite counter terms

In the algebraic formulation of QFT à la causal perturbation theory [19], one directly formulates the renormalized theory in terms of *time-ordered products* or *renormalization schemes* T which are defined to satisfy a set of physically reasonable renormalization conditions. These quantities are defined in the off-shell algebra $\hat{\mathbf{W}}_0$ which turns out to be more suitable for perturbation theory than $\hat{\mathcal{F}}_0$. In the next step, one constructs the algebra of interacting quantum fields $\hat{\mathbf{W}}_I = \hat{\mathbf{W}}_0[[\lambda]]$ as formal power series in λ with coefficients in $\hat{\mathbf{W}}_0$ which is reviewed in the next section 3.2.

Definition 3 (Renormalization schemes or time-ordered products). A renormalization scheme T is a collection of multi-linear maps

$$T_n : \mathbf{P}(M)^{\otimes n} \rightarrow D'(M^n; \hat{\mathbf{W}}_0), \quad (31)$$

that is, each $T_n(\mathcal{O}_1(x_1) \otimes \dots \otimes \mathcal{O}_n(x_n))$ is a $\hat{\mathbf{W}}_0$ -valued distribution in n space-time variables x_1, \dots, x_n . It satisfies the following axioms (renormalization conditions)

T1) Locality and covariance. For locally isometric spacetimes (M, g) and (M', g') , it holds

$$\alpha_\psi \circ T_g = T_{g'} \circ \otimes \psi_*. \quad (32)$$

Here, $\psi : M \rightarrow M'$ is a causality preserving isometric embedding, i.e. $\psi^* g' = g$, and α_ψ is the corresponding canonical homomorphism

$$\alpha_\psi : \hat{\mathbf{W}}_0(M, g) \rightarrow \hat{\mathbf{W}}_0(M', g'), \quad \alpha_\psi(F(u)) = F(\psi_* u), \quad (33)$$

with $\psi_* : \mathbf{P}(M) \rightarrow \mathbf{P}(M')$ being the natural push-forward map.

T2) Scaling. Each T_n has a poly-homogeneous scaling behavior under $g \mapsto \mu^2 g$. More precisely, let

$$T_g^{(\mu)}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n) = \mu^{d_1 + \cdots + d_n} \sigma_\mu^{-1} \circ T_{\mu^2 g}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n), \quad (34)$$

where d_i is the dimension of \mathcal{O}_i , and $\sigma_\mu : \hat{\mathbf{W}}_0(M, g) \rightarrow \hat{\mathbf{W}}_0(M, \mu^2 g)$ is an $*$ -isomorphism defined by $\sigma_\mu(\cdot : \mathcal{O}_i :_\omega) = \mu^{d_i} : \mathcal{O}_i :_\omega$. $T_g^{(\mu)} \equiv T^{(\mu)}$ is a new renormalization scheme in the algebra $\hat{\mathbf{W}}_0(M, g)$, and the axiom says that $T_n^{(\mu)}$ is a polynomial in $\log \mu$ of order at most n .

T3) Microlocal spectrum condition. The wave-front set of each $T_n(\mathcal{O}_1(x_1) \otimes \cdots \otimes \mathcal{O}_n(x_n))$ is bounded by $WF(T_n) \subset C_T(M, g)$, where

$$C_T(M, g) = \{(x_1, k_1; \dots; x_n, k_n) \in T^*M^n \setminus 0 \mid \exists G(p) \text{ with vertices } x_1 \dots x_n \in M, \\ \text{and } k_i = \sum_{\{e \mid s(e)=i\}} p_e - \sum_{\{e \mid t(e)=i\}} p_e, \forall i = 1, \dots, n\}. \quad (35)$$

In this expression, $G(p)$ in a graph embedded in M whose vertices are points $x_1 \dots x_n \in M$, and whose edges are oriented null geodesics. P_e is the coparallel, cotangent, covector field of e . $s(e) = i$, and $t(e) = j$ are respectively the source and target of the edge e connecting x_i and x_j with $i < j$. It is required that p_e is future/past directed if $x_{s(e)} \notin J^\pm(x_{t(e)})$.

T4) Smoothness and Analyticity¹. Each T_n is a smooth and analytic functional of the metric g .

T5) Graded symmetry. Each T_n is graded symmetric under a permutation of the tensor factors.

T6) Unitarity. Renormalization schemes are unitary in the following sense

$$[T_n(\otimes_i \mathcal{O}_i(x_i)^*)]^\dagger = \sum_{I_1 \sqcup \cdots \sqcup I_j = \underline{n}} (-1)^{n+j} T_{|I_1|}(\otimes_{i \in I_1} \mathcal{O}_i(x_i)) \star \cdots \star T_{|I_j|}(\otimes_{i \in I_j} \mathcal{O}_j(x_j)), \quad (36)$$

where I_1, \dots, I_j are pairwise disjoint subsets of $\underline{n} = \{1, \dots, n\}$.

T7) Causal factorization. For $x_1, \dots, x_i \cap J^-(\{x_{i+1}, \dots, x_n\}) = \emptyset$, it holds

$$T_n(\mathcal{O}_1(x_1) \otimes \cdots \otimes \mathcal{O}_n(x_n)) \\ = T_i(\mathcal{O}_1(x_1) \otimes \cdots \otimes \mathcal{O}_i(x_i)) \star T_{n-i}(\mathcal{O}_{i+1}(x_{i+1}) \otimes \cdots \otimes \mathcal{O}_n(x_n)). \quad (37)$$

T8) Commutator. The commutator of each T_n with a basic field $\Phi(x)$ is implemented as

$$[T_n(\mathcal{O}_1(x_1) \otimes \cdots \otimes \mathcal{O}_n(x_n)), \Phi^i(x)] \\ = i\hbar \sum_{k=1}^n T_n(\mathcal{O}_1(x_1) \otimes \cdots \otimes \int_M \Delta^{ij}(x, y) \frac{\delta \mathcal{O}_k(x_i)}{\delta \Phi^j(y)} \otimes \cdots \otimes \mathcal{O}_n(x_n)). \quad (38)$$

T9) Free field equation The free field equations, $\frac{\delta S_0}{\delta \Phi(x)} = 0$, is implemented in the following sense

$$T_{n+1}(\frac{\delta S_0}{\delta \Phi(x)} \otimes \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n) = i\hbar \sum_{i=1}^n T_n(\mathcal{O}_1 \otimes \cdots \otimes \frac{\delta \mathcal{O}_i(x_i)}{\delta \Phi(x)} \otimes \cdots \otimes \mathcal{O}_n). \quad (39)$$

¹It is shown in [23] that the analyticity assumption can be dropped.

T10) Action Ward identity² T_n commutes with derivatives, i.e.

$$d_{x_i} T_n(\mathcal{O}_1(x_1) \otimes \cdots \otimes \mathcal{O}_n(x_n)) = T_n(\mathcal{O}_1(x_1) \otimes \cdots \otimes d_{x_i} \mathcal{O}_i(x_i) \otimes \cdots \otimes \mathcal{O}_n(x_n)). \quad (40)$$

The crucial fact about the renormalization schemes, proved in [9], is that they exist and are unique up to a well-characterized, local and covariant “renormalization ambiguity”. This existence and uniqueness theorem is precisely formulated in the following.

Theorem 4 (The main theorem of renormalization theory [9], [8]). *Renormalization schemes satisfying the axioms of definition 3 exist. Let T and \tilde{T} be two renormalization schemes which satisfy those axioms. Then they are related via*

$$\tilde{T}_n(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n) = \sum_{I_0 \cup \cdots \cup I_r \subset \underline{n}} T_{r+1} \left(\bigotimes_k \left(\frac{\hbar}{i} \right)^{|I_k|} D_{|I_k|} \left(\bigotimes_{i \in I_k} \mathcal{O}_i \right) \otimes \bigotimes_{j \in I_0} \mathcal{O}_j \right), \quad (41)$$

where the sum runs over all partitions $I_0 \cup \cdots \cup I_r$ of the set $\underline{n} = \{1, \dots, n\}$ into pairwise disjoint non-empty subsets, and where $D = (D_n)_{n \geq 1}$ is a hierarchy of maps

$$D_n : \mathbf{P}(M)^{\otimes n} \rightarrow \mathbf{P}^{k_1/\dots/k_n}(M)[[\hbar]], \quad (42)$$

where $\mathbf{P}^{k_1/\dots/k_n}(M)[[\hbar]]$ is the space of all distributional local and covariant functionals supported on the total diagonal, and are a k_i -form in the i -th argument x_i , for all $i = 1, \dots, n$, and satisfy

- D1)** $D_n(\mathcal{O}_1(x) \otimes \cdots \otimes \mathcal{O}_n(x))$ is of order $O(\hbar)$ if all \mathcal{O}_i are of order $O(\hbar^0)$,
- D2)** Each D_n is locally, and covariantly constructed out of g , and is an analytic functional of g ,
- D3)** Each $D_n(\mathcal{O}_1(x) \otimes \cdots \otimes \mathcal{O}_n(x))$ is supported on the total diagonal

$$\Delta_n = \{(x, x, \dots, x) | x \in M\} \subset M^n, \quad (43)$$

- D4)** Each D_n is graded symmetric,
- D5)** The maps D_n are real $D_n(\mathcal{O}_1(x) \otimes \cdots \otimes \mathcal{O}_n(x))^* = D_n(\mathcal{O}_1^*(x) \otimes \cdots \otimes \mathcal{O}_n^*(x))$,
- D6)** Each D_n satisfies the dimension constraint

$$(\mathbb{N}_d + \Delta_s) D_n(\mathcal{O}_1(x) \otimes \cdots \otimes \mathcal{O}_n(x)) = \sum_{i=1}^n D_n(\mathcal{O}_1(x) \otimes \cdots \otimes \mathbb{N}_d \mathcal{O}_i(x_i) \otimes \cdots \otimes \mathcal{O}_n(x)), \quad (44)$$

where \mathbb{N}_d is the dimension counter operator, and Δ_s is the scaling degree of distributions,

- D7)** Derivatives can be pulled into D_n ,

$$d_{x_i} D_n(\mathcal{O}_1(x_1) \otimes \cdots \otimes \mathcal{O}_n(x_n)) = D_n(\mathcal{O}_1(x_1) \otimes \cdots \otimes d_{x_i} \mathcal{O}_i(x_i) \otimes \cdots \otimes \mathcal{O}_n(x_i)). \quad (45)$$

Conversely, if D satisfies **D1** - **D7**, then any \tilde{T} defined by (41) is a new renormalization scheme.

²Note that axiom **T11** implies $T_n(\cdots \otimes \int f \wedge d\mathcal{O} \otimes \cdots) = T_n(\cdots \otimes - \int df \wedge \mathcal{O} \otimes \cdots)$, which in turn means that each T_n may be equivalently viewed as a map $T_n : \mathcal{A}^{\otimes n} \rightarrow \tilde{\mathbf{W}}_0$, on the space of local action functionals.

Conservation of free BRST current

Similar to the split of the extended action (21), we also split the BRST current

$$\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_{\text{int}}. \quad (46)$$

We will now show that the quantized $\mathbf{J}_0 \in \hat{\mathcal{F}}_0$, is indeed conserved. This is a prerequisite for formulating the Ward identity (65) which in turn will imply the conservation of the full current $\mathbf{J}_I \in \hat{\mathcal{F}}_I$.

The time-ordered products $T_1(\mathcal{O}(x))$ with one factor which satisfy the local and covariance property of T_n are constructed [8] as local Wick powers $:\mathcal{O}(x):_H$ with respect to a Hadamard parametrix $H^{ij}(x, y)$. A Hadamard parametrix is a distribution defined in a convex normal neighbourhood $U \times U$ of the diagonal in $M \times M$, which is a bi-solution of the free equations of motion modulo $C^\infty(M \times M)$, with a specific wave-front set (see [21]), and satisfies $\text{Im } H^{ij}(x, y) = \frac{1}{2}\Delta^{ij}(x, y)$. Explicitly,

$$H^{ij}(x, y) = k_{IJ} \otimes \begin{pmatrix} H^{\mu\nu}(x, y) & 0 & 0 & -i\delta_y H^{\mu\nu}(x, y) \\ 0 & 0 & iH(x, y) & 0 \\ 0 & -iH(x, y) & 0 & 0 \\ -i\delta_x H^{\mu\nu}(x, y) & 0 & 0 & 0 \end{pmatrix}, \quad (47)$$

where $H(x, y)$, $H^{\mu\nu}(x, y)$ are scalar and vector Hadamard parametrices [17], given by

$$H(x, y) = \frac{1}{2\pi^2} \left(\frac{u(x, y)}{\sigma + it0} + v(x, y) \log(\sigma + it0) \right), \quad (48)$$

$$H_{\mu\nu}(x, y) = \frac{1}{2\pi^2} \left(\frac{u_{\mu\nu}(x, y)}{\sigma + it0} + v_{\mu\nu}(x, y) \log(\sigma + it0) \right). \quad (49)$$

In the above expressions, $\sigma(x, y)$ is the signed squared geodesic distance between $(x, y) \in U \times U$ and $u, v, u_{\mu\nu}, v_{\mu\nu}$ are smooth functions on $U \times U$ which are determined by requiring H^s and H^v to be bi-solutions of the equations of motion.

The important fact about the local Wick powers, is that $:\mathcal{O}(x):_H$ differs from $:\mathcal{O}(x):_\omega$ only by a smooth function valued in $\hat{\mathbf{W}}_0$, see e.g. [1] appendix E.

Theorem 5. *In pure Yang-Mills theory,*

1. *The local and covariant free BRST current $:\mathbf{J}_0:_H \in \hat{\mathcal{F}}_0$ is conserved,*
2. *the free BRST charge $:Q_0:_H = \int_M \gamma \wedge : \mathbf{J}_0:_H$ (c.f. definition (18)) squares to zero.*

Proof. The divergent of the free part of the classical current \mathbf{J}_0 takes the form

$$d\mathbf{J}_0 = d * dA^I \wedge dC_I - idB^I * dC_I - iB^I d * dC_I. \quad (50)$$

Since in (47) there is no “contraction” between either A^I and C^I , or B^I and C^I , we have $:d\mathbf{J}_0(x):_H = d\mathbf{J}_0(x)$ (the classical current) which vanishes on-shell. For the same reason, we have

$$:Q_0:_H^2 = :Q_0:_H \star :Q_0:_H = 0. \quad (51)$$

□

Remark 6. *Conservation of the free BRST current (and hence the existence of Q_0) as well as $Q_0^2 = 0$ is a prerequisite for formulating the ward identity (65), from which our main results follows. Although the above proof of those requirements seems to be specific to the Yang-Mills current, it turns out [2], [3], that this “no-contraction” behavior of $:d\mathbf{J}_0:_H$ and $H^{ij}(x, y)$ is a universal one.*

3.2 Renormalized interacting quantum fields

Having defined the algebra of free quantum fields $\hat{\mathbf{W}}_0$ associated to the enlarged classical theory with action functional $\hat{S} = \hat{S}_0 + I$, and the renormalization schemes on $\hat{\mathbf{W}}_0$, we now turn to defining the \star -algebra of interacting quantum fields $\hat{\mathbf{W}}_I$. The building blocks of the algebraic approach to interacting QFT are called *interacting fields* $\mathcal{O}(x)_I \in \hat{\mathbf{W}}_I$, associated with a given classical observable $\mathcal{O}(x)$, $x \in M$ and an interaction $I = \int \mathbf{L}_{\text{int}}$. However in our local approach, we first consider a causal domain $O \subset M$ containing x , and consider a cutoff interaction $\int f \mathbf{L}_{\text{int}}$ where $f = 1$ on an open neighbourhood of O . In other words, we first define the interacting fields $\mathcal{O}(x)_L$, $x \in O$, and then obtain $\mathcal{O}(x)_I$, $x \in M$ as the *algebraic adiabatic limit* of $\mathcal{O}(x)_L$, where $f \rightarrow 1$ on the whole M , in the precise manner explained at the end of this part.

Let us denote by $T(e_{\otimes}^F)$ the *generating functional* of all T_n 's, that is

$$T(e_{\otimes}^F) = \sum_n \frac{1}{n!} T_n(F^{\otimes n}). \quad (52)$$

We begin by defining the interacting analogue of the time ordered products.

Definition 7. *Given a renormalization scheme, T_n , the interacting time-ordered product of n local functionals $\mathcal{O}_1(x_1), \dots, \mathcal{O}_n(x_n)$, associated with the cutoff interaction L is the map*

$$T_{L,n} : \mathbf{P}(M)^{\otimes n} \rightarrow D'(M^n; \hat{\mathbf{W}}_0[[\lambda]]) \quad (53)$$

defined by

$$\begin{aligned} T_{L,n}(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n) &:= T(e_{\otimes}^{iL/\hbar})^{-1} \star T(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\ &= \frac{d^n}{d\tau_1 \dots d\tau_n} T(e_{\otimes}^{iL/\hbar})^{-1} \star T(e_{\otimes}^{iL/\hbar + \tau_1 \mathcal{O}_1 + \dots + \tau_n \mathcal{O}_n})|_{\tau_i=0}. \end{aligned} \quad (54)$$

The interacting time-ordered product can be equivalently written using the *retarded products*

$$R_{n,k} : \mathbf{P}^{\otimes(n+k)} \rightarrow D'(M^n; \hat{\mathbf{W}}_0[[\lambda]]), \quad (55)$$

defined via

$$T_{L,n}(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n) =: \sum_{k=0} \frac{i^k}{k! \hbar^k} R_{n,k}(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n; L^{\otimes k}) \equiv R(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n; e^{iL/\hbar}). \quad (56)$$

They are called retarded because of the following support property

$$\begin{aligned} \text{supp } R_{n,k}(\mathcal{O}_1(x_1) \otimes \dots \otimes \mathcal{O}_n(x_n); \mathcal{O}_{n+1}(x_{n+1}) \otimes \dots \otimes \mathcal{O}_{n+k}(x_{n+k})) \\ \in \{(x_1, \dots, x_{n+k}) | \{x_1, \dots, x_n\} \in J^-(\{x_{n+1}, \dots, x_{n+k}\})\}. \end{aligned} \quad (57)$$

For the particular case of one local field, $\mathcal{O}(x)$, the interacting time-ordered product is called an *interacting field* under interaction L and is denoted $\mathcal{O}(x)_L \equiv T_{L,1}(\mathcal{O})$. The linear span of all $T_{L,n}(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n)$ equipped with \star form the algebra $\hat{\mathbf{W}}_L$. It then follows that all $T_{L,n}$ with $n \geq 2$ can be written in terms of the product of interacting fields and retarded products. For instance, for $n = 2$, we obtain

$$T_{L,2}(\mathcal{O}_1 \otimes \mathcal{O}_2) = (\mathcal{O}_2)_L \star (\mathcal{O}_1)_L + R(\mathcal{O}_1; \mathcal{O}_2 \otimes e_{\otimes}^{iL/\hbar}), \quad (58)$$

where $R(\mathcal{O}_1; \mathcal{O}_2 \otimes e_{\otimes}^{iL/\hbar}) = \frac{d}{d\tau} \Big|_{\tau=0} R(\mathcal{O}_1; e_{\otimes}^{iL/\hbar + \tau \mathcal{O}_2})$.³ Using relation (58) and the symmetry property of T_n , it is easy to prove the GLZ (Glaser-Lehmann-Ziemmermann) formula:

$$[\mathcal{O}_1(x)_L, \mathcal{O}_2(y)_L] = R(\mathcal{O}_1(x); \mathcal{O}_2(y) \otimes e_{\otimes}^{iL/\hbar}) - R(\mathcal{O}_2(y); \mathcal{O}_1(x) \otimes e_{\otimes}^{iL/\hbar}). \quad (59)$$

From the GLZ formula, together with (57), it follows that

$$[\mathcal{O}_1(x)_L, \mathcal{O}_2(y)_L] = 0, \quad \text{if } x \text{ and } y \text{ are causally separated.} \quad (60)$$

Algebraic adiabatic limit

So far, we have shown how to (perturbatively) construct the interacting fields $\mathcal{O}(x)_L$ with a local interaction $L = \int f \mathbf{L}_{\text{int}}$, in a causal domain $O \subset M$ which is contained in the region where the cutoff $f = 1$. In fact, it can be shown [7] that changing the cutoff function to f' which coincide with f on some neighbourhood of O changes \mathcal{O}_L via a unitary transformation $V_{f,f'}$:

$$\mathcal{O}_{L'} = V_{f,f'} \star \mathcal{O}_L \star V_{f,f'}^{-1}, \quad (61)$$

where $L' = \int f' \mathbf{L}_{\text{int}}$. Indeed, while the naive adiabatic limit $f \rightarrow 1$ for \mathcal{O}_L does not exist, the insertion of unitaries in (61) allows for defining the true interacting field \mathcal{O}_I . Similarly, one can define $T_I(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)$. With this, we define $\hat{\mathbf{W}}_I(M, g) = \text{Alg}\{T_I(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n), \star\}$, and

$$\hat{\mathcal{F}}_I(M, g) = \hat{\mathbf{W}}_I(M, g) / \mathcal{J}_0. \quad (62)$$

The existence of the algebraic adiabatic limit implies that it is enough to choose cutoff functions which are equal 1, in a neighbourhood of O . Then, the cutoff can be sent to 1 on the entire space-time. Put differently, in order to prove statements about $\mathcal{O}_I(x)$, it suffices to work with the cutoff interaction L where $f = 1$ in a sufficiently large neighbourhood containing x . Indeed, our main result (112), holds for all points $x_1, \dots, x_n \in O$ which lie in the region on which $f = 1$.

Remark 8.

For $L = \int f \mathbf{L}_{\text{int}}$, formally $\mathcal{S} = T(e_{\otimes}^{iL/\hbar})$ (the “local S -matrix”) tends to the S -matrix of the theory in the limit where the cutoff is sent to 1 on the entire space-time. Note, however, that \mathcal{S} is not an element of $\hat{\mathbf{W}}_L$, and therefore the existence of the “true S -matrix” of the theory, cannot be established by the above type of arguments which leads to the existence of the algebraic adiabatic limit for the interacting fields in $\hat{\mathbf{W}}_L$. Whether an in what sense such an adiabatic limit exists is related to the infra-red properties of the S -matrix whose existence is a non-trivial and difficult task to establish even in Minkowski space-time. Here, we are not concerned with this issue, and we explicitly keep the IR cutoff in all the constructions of the renormalized theory. Therefore in our completely local approach, we disentangle the formulation of the renormalization of quantum fields which is a short distance, and hence UV, issue from the IR issues which does not show up for \mathcal{S} .

³Note the difference between $R(\mathcal{O}_1; \mathcal{O}_2 \otimes e_{\otimes}^{iL/\hbar})$ and $R(\mathcal{O}_1 \otimes \mathcal{O}_2; e_{\otimes}^{iL/\hbar})$: in the latter, both \mathcal{O}_1 and \mathcal{O}_2 are separated from interaction by the semi-column, while in the former \mathcal{O}_1 is separated from \mathcal{O}_2 and interaction by the semi-column. Their difference is indeed $\mathcal{O}_{2L} \star \mathcal{O}_{1L}$.

3.3 Ward identities and conserved BRST current

The preservation of the BRST symmetry at quantum level takes the form of a further renormalization condition imposed on T ; the ‘‘Ward identity’’ [1]. In order to formulate this identity we need to extend the action of \hat{s}_0 on the algebra \mathbf{W}_0 . In fact, using the consistency relations

$$\nabla_\mu \Delta^{\mu\nu}(x, y) = -\nabla^\nu \Delta(x, y), \quad \nabla_\nu \Delta^{\mu\nu}(x, y) = -\nabla^\mu \Delta(x, y), \quad (63)$$

it follows that \hat{s}_0 can be consistently extended to $\hat{\mathbf{W}}_0$ as a graded derivation, that is, it satisfies the Leibniz rule

$$\hat{s}_0(\mathcal{O}_1 \star \cdots \star \mathcal{O}_n) = \sum_k (-1)^{\sum_{l < k} \epsilon_l} \mathcal{O}_1 \star \cdots \star \hat{s}_0 \mathcal{O}_i \star \cdots \star \mathcal{O}_n, \quad (64)$$

and preserves the commutation relations (29), (30) in $\hat{\mathbf{W}}_0$.

Now, for a given renormalization scheme T , and for $F = \int f \wedge \mathcal{O}$ for all $\mathcal{O} \in \mathbf{P}^p(M)$, $f \in \Omega_0^{4-p}(M)$, the Ward identity takes the form⁴

$$[Q_0, T(e_{\otimes}^{iF/\hbar})] = -\frac{1}{2}T\left((\hat{S}_0 + F, \hat{S}_0 + F) \otimes e_{\otimes}^{iF/\hbar}\right), \quad \text{mod } \mathcal{J}_0. \quad (65)$$

In this formula, $Q_0 \equiv T_1(Q_0) =: Q_0 :_H$ is the free BRST charge defined above equation (50), and $(-, -)$ is the anti-bracket bracket defined in (10). Note that the graded derivation $[Q_0, -]$ is nilpotent since $[Q_0, [Q_0, -]] = \frac{1}{2}[Q_0^2, -] = 0$, by equation (51).

The Ward identity (65) is, however, in general violated by a potential *anomaly* term. To define it properly, we express the off-shell violation of (65), where $[Q_0, -]$ is replaced by $i\hbar\hat{s}_0$ which acts non-trivially also on anti-fields, in the following theorem.

Theorem 9 (anomalous Ward identity [1]). *Let $F = \int f \wedge \mathcal{O}$ for all $\mathcal{O} \in \mathbf{P}^p(M)$ and $f \in \Omega_0^{4-p}(M)$. Then for a chosen renormalization scheme T it holds*

$$\hat{s}_0 T(e_{\otimes}^{iF/\hbar}) = \frac{i}{2\hbar}T\left((\hat{S}_0 + F, \hat{S}_0 + F) \otimes e_{\otimes}^{iF/\hbar}\right) + \frac{i}{\hbar}T(A(e_{\otimes}^F) \otimes e_{\otimes}^{iF/\hbar}). \quad (66)$$

The second term in the right hand side defines the anomaly $A(e_{\otimes}^F) = \sum_n \frac{1}{n!}A_n(F^{\otimes n})$, where each A_n is a map

$$A_n : \mathbf{P}(M)^{\otimes n} \rightarrow \mathbf{P}^{k_1/\dots/k_n}(M)[[\hbar]], \quad (67)$$

with properties

- A1)** $A_n(\mathcal{O}_1(x) \otimes \cdots \otimes \mathcal{O}_n(x))$ is of order $O(\hbar)$ if all \mathcal{O}_i are of order $O(\hbar^0)$,
- A2)** Each A_n is locally, and covariantly constructed out of g , and is an analytic functional of g ,
- A3)** Each $A_n(\mathcal{O}_1(x) \otimes \cdots \otimes \mathcal{O}_n(x))$ is supported on the total diagonal Δ_n ,
- A4)** Each A_n increases the ghost number by one unit,

⁴Note that for the renormalization schemes T_n to exist, their arguments have to be local functionals, or cutoff integrated functionals. However, in the expression (65), $\hat{S}_0 = \int \mathbf{L}_0$ need not be cut off, since, on account of $(\hat{S}_0, \hat{S}_0) = 0$, it appears only in the form $(\hat{S}_0 + F, \hat{S}_0 + F) = 2(\hat{S}_0, F) + (F, F)$, and $(\hat{S}_0, F) = \int f(x) \frac{\delta \mathbf{L}_0}{\delta \Phi(x)} \frac{\delta \mathcal{O}}{\delta \Phi^\dagger(x)} + \frac{\delta \mathbf{L}_0}{\delta \Phi^\dagger(x)} \frac{\delta \mathcal{O}}{\delta \Phi(x)}$ is cut off with f .

A5) Each A_n is graded symmetric,

A6) The maps A_n are real $A_n(\mathcal{O}_1(x) \otimes \cdots \otimes \mathcal{O}_n(x))^* = A_n(\mathcal{O}_1^*(x) \otimes \cdots \otimes \mathcal{O}_n^*(x))$,

A7) Each A_n satisfies the dimension constraint

$$(\mathcal{N}_d + \Delta_s) A_n(\mathcal{O}_1(x) \otimes \cdots \otimes \mathcal{O}_n(x)) = \sum_{i=1}^n A_n(\mathcal{O}_1(x) \otimes \cdots \mathcal{N}_d \mathcal{O}_i(x_i) \otimes \cdots \otimes \mathcal{O}_n(x)), \quad (68)$$

A8) Derivatives can be pulled into A_n ,

$$d_{x_i} A_n(\mathcal{O}_1(x_1) \otimes \cdots \otimes \mathcal{O}_n(x_n)) = A_n(\mathcal{O}_1(x_1) \otimes \cdots \otimes d_{x_i} \mathcal{O}_i(x_i) \otimes \cdots \otimes \mathcal{O}_n(x_n)), \quad (69)$$

A9) $A_n(\mathcal{O}_1(x) \otimes \cdots \otimes \mathcal{O}_n(x)) = 0$ if one entry contains no dynamical field.

Note that the anomalous Ward identity defines the anomaly $A(e_{\otimes}^F)$ as a map

$$A_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}[[\hbar]], \quad (70)$$

on the space of local actions. This is indeed possible due to the property (69) (see footnote 3).

The anomalous Ward identity is an identity for generating functionals $T(e_{\otimes}^F)$ and $A(e_{\otimes}^F)$. Consider n local functionals F_1, \dots, F_n . Then, (66) at order n reads

$$\begin{aligned} \hat{s}_0 T_n(F_1 \otimes \cdots \otimes F_n) &= \sum_{k=0} T_n(F_1 \otimes \cdots \otimes \hat{s}_0 F_k \otimes \cdots \otimes F_n) \\ &+ \frac{\hbar}{i} \sum_{I_2} T_{n-1}((F_i, F_j)_{i,j \in I_2} \otimes \bigotimes_{k \in I_2^c} F_k) \\ &+ \sum_I \left(\frac{\hbar}{i}\right)^{|I|-1} T_{n-|I|+1}(A_{|I|}(\bigotimes_{i \in I} F_i) \otimes \bigotimes_{j \in I^c} F_j), \end{aligned} \quad (71)$$

where I is a non-empty partition of the set $\{1, 2, \dots, n\}$, I^c is the complement partition and $|I_2| = 2$. The correct signs needed in the above expression where F_i are of arbitrary Grassmann parity are given in appendix A, equation (169).

Remark 10. To be well-defined, the anomaly has to take either local functionals, $\mathcal{O}(x)$, or integrated functionals with compactly supported test function $F = \int f \mathcal{O}$. However, throughout this work, we encounter expressions where the cut off function f is sent to 1 over the whole space-time. To clarify when this is possible and when it is not, consider the following situations

1. **When A_n does not appear in the argument of T_n :** Contrary to time-ordered products which take value in the algebra $\hat{\mathbf{W}}_0$, the anomalies are local expressions in the dynamical and background fields which are supported on the total diagonal. For instance, one can calculate $A_2(\phi(x) \otimes \phi(y)) = P(\Phi(x), \Phi_{bg}(x)) \delta(x, y)$ where $P(\Phi, \Phi_{bg})$ is a local functional in dynamical fields Φ , and background fields Φ_{bg} with dimension -2 . Thus, for $F = \int f \phi$ we have

$$\begin{aligned} A_2(F \otimes F) &= \int_{M^2} dx dy f(x) f(y) P(\Phi(x), \Phi_{bg}(x)) \delta(x, y) \\ &= \int_M dx f(x)^2 P(\Phi(x), \Phi_{bg}(x)), \end{aligned}$$

i.e. $A_2(F \otimes F)$ is reduced to an integral only over M with one cutoff, by the virtue of $\delta(x, y)$. However, we can also set $f = 1$ in the above integral since we are taking the (off-shell) dynamical configurations to be compactly supported.⁵ Note that although the background configurations may not be compactly supported, as long as one compactly supported dynamical field appears in such expressions, the integral exists.⁶

2. **When A_n appears in the argument of T_n :** In this case, we are not allowed to set $f = 1$, even when the field configurations are taken to be compactly supported, and as mentioned in remark 8 for time-ordered products to exist it is necessary that T_n takes compactly supported functionals. Therefore, for example for the case $L = \int f \mathbf{L}_{\text{int}}$, the anomaly in (66) cannot be written as $T(e_{\otimes}^L \otimes A(e_{\otimes}^I))$ (with $I = \int \mathbf{L}_{\text{int}}$) and to remove the anomaly, one has to argue that there exists a renormalization scheme in which $A(e_{\otimes}^L) = 0$ (we will briefly review this argument below). Nevertheless, in expressions of the form $A(e_{\otimes}^F \otimes \mathcal{O}(x))$ which contains at least one local polynomial $\mathcal{O}(x)$, we can safely remove the cutoff since each $A_n(F \otimes \cdots \otimes F \otimes \mathcal{O}(x))$ with $n-1$ cutoffs contains an n -fold delta function $\delta(x_1, \dots, x_n)$ (or derivatives of δ). Hence, expressions such as $T(e_{\otimes}^L \otimes A(e_{\otimes}^I \otimes \mathcal{O}(x)))$ are well-defined.

Consistency conditions and removal of anomalies

The study of anomaly turns out to reduce to a cohomological problem. In fact, triviality of the cohomology class $H_1^4(\hat{s}|d, M)$, which contains potential anomalies, leads to the Ward identity (65). Let us briefly review the argument. To this end (and for the proof of several statements in the next section), we need the following lemma.

Lemma 11. Let $(\hat{A}_n)_{n \geq 1}$ be a family of operators

$$\hat{A}_n : \mathbf{P}(M)^{\otimes n} \rightarrow \mathbf{P}(M)[[\hbar]], \quad (72)$$

with $\hat{A}_n(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n) = A(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^I)$, and set $\hat{A}_0 = A(e_{\otimes}^I)$. Then, we have

$$\begin{aligned} & \hat{s}\hat{A}_n(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n) + \sum_{\sigma} (\mathcal{O}_{\sigma(1)}, \hat{A}_{n-1}(\mathcal{O}_{\sigma(2)} \otimes \cdots \otimes \mathcal{O}_{\sigma(n)})) \\ & + \sum_{i=1}^n \hat{A}_n(\mathcal{O}_1 \otimes \cdots \otimes \hat{s}\mathcal{O}_i \otimes \cdots \otimes \mathcal{O}_n) \\ & + \sum_{I_2} \hat{A}_{n-1}((\mathcal{O}_i, \mathcal{O}_j)_{i,j \in I_2} \otimes \bigotimes_{k \in I_2^c} \mathcal{O}_k) \\ & + \sum_I \hat{A}_{n-|I|+1} \left(\hat{A}_{|I|} \left(\bigotimes_{i \in I} \mathcal{O}_i \right) \otimes \bigotimes_{j \in I^c} \mathcal{O}_j \right) \\ & + \hat{A}_{n+1}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes A(e_{\otimes}^I)) = 0, \end{aligned} \quad (73)$$

where the sum runs over all non-empty partitions I of the set $\{1, 2, \dots, n\}$, I^c is the complement partition, $|I_2| = 2$ and $\sigma \in S_n$.

⁵This is precisely the same situation for the classical action $S = \int_M \mathbf{L}(\Phi, \Phi_{\text{bg}})$ which exist only on off-shell compactly supported dynamical field configurations $\Phi(x)$.

⁶Since anomaly increases the ghost number by 1, in practice always a dynamical field (ghost) would appear in the integrand.

Proof. Applying \hat{s}_0 on both sides of (66) and using $\hat{s}_0^2 = 0$, it leads to the following “consistency condition” derived in [1] proposition 4:

$$(\hat{S}_0 + F, A(e_\otimes^F)) + A\left(\left(\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + A(e_\otimes^F)\right) \otimes e_\otimes^F\right) = 0. \quad (74)$$

This identity is valid for all functionals F . According to remark 10, we can set $F = I$, and using $(\hat{S}_0 + I, \hat{S}_0 + I) = 0$ we obtain

$$\hat{q}A(e_\otimes^I) = 0, \quad (75)$$

which is (73) for $n = 1$. To derive the identity (73), we similarly replace F with $F + \tau_1 \mathcal{O}_1 + \dots \tau_n \mathcal{O}_n$ in (74), differentiate with respect to $\tau_1 \dots \tau_n$ and set $\tau_i = 0$. This procedure, together with the following relations

$$\frac{d}{d\tau} \left(\frac{1}{2}(\hat{S}_0 + F + \tau \mathcal{O}, \hat{S}_0 + F + \tau \mathcal{O}) + A(e_\otimes^{F+\tau \mathcal{O}}) \right) \Big|_{\tau=0} = (\hat{S}_0 + F, \mathcal{O}) + A(\mathcal{O} \otimes e_\otimes^F), \quad (76)$$

$$\frac{d}{d\tau} \left((\hat{S}_0 + F + \tau \mathcal{O}_2, \mathcal{O}_1) + A(\mathcal{O}_1 \otimes e_\otimes^{F+\tau \mathcal{O}_2}) \right) \Big|_{\tau=0} = (\mathcal{O}_1, \mathcal{O}_2) + A(\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes e_\otimes^F), \quad (77)$$

$$\frac{d}{d\tau} A(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_k \otimes e_\otimes^{F+\tau \mathcal{O}_{k+1}}) \Big|_{\tau=0} = A(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_{k+1} \otimes e_\otimes^F), \quad (78)$$

leads, after a straightforward but lengthy calculation, to

$$\begin{aligned} & (\hat{S}_0 + F, A(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_\otimes^F)) \\ & + \sum_{\sigma} (\mathcal{O}_{\sigma(1)}, A(\mathcal{O}_{\sigma(2)} \otimes \dots \otimes \mathcal{O}_{\sigma(n)} \otimes e_\otimes^F)) \\ & + \sum_{i=1}^n A(\mathcal{O}_1 \otimes \dots \otimes (\hat{S}_0 + F, \mathcal{O}_i) \otimes \dots \otimes \mathcal{O}_n \otimes e_\otimes^F) \\ & + \sum_{I_2} A((\mathcal{O}_i, \mathcal{O}_j)_{i,j \in I_2} \otimes \bigotimes_{k \in I_2^c} \mathcal{O}_k \otimes e_\otimes^F) \\ & + \sum_I A\left(A\left(\bigotimes_{i \in I} \mathcal{O}_i \otimes e_\otimes^F\right) \otimes \bigotimes_{j \in I^c} \mathcal{O}_j \otimes e_\otimes^F\right) \\ & + A(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes \left(\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + A(e_\otimes^F)\right) \otimes e_\otimes^F) = 0. \end{aligned} \quad (79)$$

Now setting $F = I$, and using $(\hat{S}_0 + I, \hat{S}_0 + I) = 0$ in the above equation, we arrive at (73). □

It is not difficult to see that, when $A(e_\otimes^I) = 0$, all the consistency conditions can be integrated into one identity for the generating functional $\hat{A}(e_\otimes^F) = \sum_{n=1} \frac{1}{n!} \hat{A}_n(F^{\otimes n})$ of \hat{A}_n , which is

$$(\hat{S}_0 + F, \hat{A}(e_\otimes^F)) + \hat{A}\left(\left(\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + \hat{A}(e_\otimes^F)\right) \otimes e_\otimes^F\right) = 0. \quad (80)$$

Let us now briefly review how in certain cases the anomaly can be removed by passing to a new renormalization scheme.

Theorem 12 ([1]). *If the cohomology ring $H_1^4(\hat{s}|d)$ is trivial, then there exists a renormalization scheme in which the anomaly $A(e_\otimes^L)$ of the anomalous Ward identity (66) for the case $F = L$ is absent.*

Sketch of proof. The proof consists of two parts: first, we show that the anomaly vanishes for $f = 1$, i.e. $A(e_\otimes^L) = 0$ and second, we show that $A(e_\otimes^L) = 0$ for all cutoff functions f .

For the first part, consider the expansion of $A(e_\otimes^L)$ in powers of \hbar

$$A(e_\otimes^L) = A^m(e_\otimes^L)\hbar^m + A^{m+1}(e_\otimes^L)\hbar^{m+1} + \dots, \quad (81)$$

for some integer $m > 0$. Then, the lowest order in the expansion of equation (75) in \hbar implies the “ \hbar -expanded consistency condition”:

$$\hat{s}A^m(e_\otimes^L) = 0. \quad (82)$$

Now we write $A^m(e_\otimes^L) = \int_M a^m(x)$ as an integral of a local four-form $a^m(x)$. Also, by property 4 of the definition of anomaly (given in theorem (9)) $a^m(x)$ has ghost number 1. Then, (82) means that $a^m(x)$ belongs to the cohomology class $H_1^4(\hat{s}|d, M)$, which is trivial. Therefore,

$$a^m(x) = \hat{s}b^m(x) + dc^m(x), \quad (83)$$

for some $b^m(x) \in \mathbf{P}_0^4(M)$ and $c^m(x) \in \mathbf{P}_1^3(M)$. We can now show that this \hat{s} -exact anomaly is absent if we pass to a specific renormalization scheme. Precisely, we need to perform the following steps: (1) chose a new scheme \tilde{T} by explicitly constructing the local finite counter terms D_n using b^m :

$$D_n^m(\mathbf{L}_1(x_1) \otimes \dots \otimes \mathbf{L}_1(x_n)) = -\hbar^m b_n^m(x_1) \delta(x_1, \dots, x_n), \quad (84)$$

where D^m is the first non-trivial term in the \hbar -expansion of $D(e_\otimes^L)$ and where we have expanded $b^m = \sum_{n>0} \frac{\lambda^n}{n!} b_n^m$, (2) rewrite the anomalous Ward identity (66) in the scheme \tilde{T} :

$$\hat{s}_0 \tilde{T}(e_\otimes^{iF/\hbar}) = \frac{i}{2\hbar} \tilde{T}((\hat{S}_0 + F, \hat{S}_0 + F) \otimes e_\otimes^{iF/\hbar}) + \frac{i}{\hbar} \tilde{T}(\tilde{A}(e_\otimes^F) \otimes e_\otimes^{iF/\hbar}). \quad (85)$$

which means that in the new scheme, $A(e_\otimes^F)$ is replaced with $\tilde{A}(e_\otimes^F)$, (3) express the new anomaly $\tilde{A}(e_\otimes^F)$ in terms of the old anomaly $A(e_\otimes^{F+D(e_\otimes^F)})$ with modified interaction $F + D(e_\otimes^F)$ ([1], equation (387)), which to lowest order in \hbar and for $f = 1$, takes the form

$$\tilde{A}^m(e_\otimes^L) = A^m(e_\otimes^L) + \hat{s}D^m(e_\otimes^L), \quad (86)$$

(4) conclude from (83) and (86) that

$$\tilde{A}^m(e_\otimes^L) = \int_M \tilde{a}^m(x) = \int_M a^m(x) - \hat{s}b^m(x) = \int_M dc^m(x) = 0. \quad (87)$$

For the second part of the proof, we use

$$0 = \tilde{A}^m(e_\otimes^L) = \sum_n \frac{\lambda^n}{n!} \int \mathcal{A}_n^m(x_1, \dots, x_n) dx_1 \dots dx_n, \quad (88)$$

as a starting point. From this equation it follows that (see [1] Lemma 9) $\mathcal{A}_n^m(x_1, \dots, x_n) = \sum_{k=1}^n d_k \mathcal{C}_{n/k}^m(x_1, \dots, x_n)$ for some $\mathcal{C}_{n/k}^m \in P^{4/\dots 3/\dots 4}(M^n)$. Using this quantities, we make a further redefinition and pass to another scheme \hat{T} by setting

$$\hat{D}_n^m(\mathbf{L}_1(x_1) \otimes \dots \otimes \mathbf{K}_1(x_k) \otimes \dots \otimes \mathbf{L}_1(x_n)) = -\hbar^m \mathcal{C}_{n/k}^m(x_1) \delta(x_1, \dots, x_n), \quad (89)$$

where $\mathbf{K}_1 \in \mathbf{P}_1^3(M)$ is defined by $\hat{s}_0 \mathbf{L}_1 = d\mathbf{K}_1$. Again, we express the lowest order in the expansion of anomaly in the scheme \hat{T} in terms of that in the scheme \tilde{T} :

$$\hat{A}^m(e_{\otimes}^L) = \tilde{A}^m(e_{\otimes}^L) + \frac{1}{2} \hat{D}((\hat{S}_0 + L, \hat{S}_0 + L) \otimes e_{\otimes}^L). \quad (90)$$

However, using (89) we have $\hat{D}^m((L, L) \otimes e_{\otimes}^L) = 0$ and $\hat{D}^m(\hat{s}_0 L \otimes e_{\otimes}^L) = -\tilde{A}^m(e_{\otimes}^L)$. This means that there exists a scheme \hat{T} in which the anomaly $\hat{A}(e_{\otimes}^L)$ vanishes to lowest order in \hbar and to all orders in λ . Proceeding by iteration in higher powers of \hbar , we conclude that the anomaly can be removed to all orders in \hbar and λ . \square

Remark 13. For the pure Yang-Mills case, $H_1^4(\hat{s}|d, M)$ is not trivial. In fact, when G is semi-simple with no abelian factors, this cohomology class is generated by the so-called “gauge anomaly” [16] of the form

$$\mathcal{A} = dC^I \wedge \left(d_{IJK} A^J \wedge dA^K - \frac{1}{12} D_{IJKL} A^J \wedge A^K \wedge A^L \right), \quad (91)$$

with d_{IJK} and D_{IJKL} being symmetric tensors in some representations of the Lie algebra \mathfrak{g} . Nevertheless, one can still argue [1] that $a^m(x)$ is the zero element in this cohomology class as follows. In Minkowski space-time, where parity is an isometry, one can argue that $a^m(x)$ is parity odd, i.e. it transforms as $a^m \mapsto -a^m$ under $\epsilon \mapsto -\epsilon$. However, the gauge anomaly (91) is evidently even under parity. Therefore $a^m(x)$ is the zero element in the cohomology. Now since anomaly is a local and covariant quantity, when it vanishes in one space-time it vanishes on all space-times.

Conservation of the interacting BRST current

We have already shown in section 3.1 that the free part \mathbf{J}_0 of the BRST current \mathbf{J} is conserved in the free theory. In this section, we review the argument in [1] which leads to the conservation of the full interacting BRST current. We show that once a renormalization scheme is chosen in which the anomaly $A(e_{\otimes}^L)$ vanishes, the interacting field corresponding to Noether current of BRST symmetry $\mathbf{J}(x)$ is conserved as a consequent of the Ward identity (65). The important point about the proof is that it holds true irrespective of the functional form of the classical BRST current $\mathbf{J}(x)$, and therefore is satisfied for a wider class of theories with local gauge symmetry which admit an appropriate BRST formulation. We first state the following proposition.

Proposition 14. Let h be a compactly supported function on M with

$$h(y) = \begin{cases} 0 & J^+(\Sigma_+) \\ 1 & \text{on an open nbh. of } \{x_1, \dots, x_n\} \subset M \\ 0 & J^-(\Sigma_-), \end{cases} \quad (92)$$

where Σ_{\pm} are Cauchy surfaces in the future/past of $\{x_1, \dots, x_n\}$. Then, we have

$$\int dy h(y) T(d\mathbf{J}(y) \otimes e_{\otimes}^{iL/\hbar}) = [Q_0, T(e_{\otimes}^{iL/\hbar})] + T\left(\frac{1}{2}(\hat{S}_0 + L, \hat{S}_0 + L) \otimes e_{\otimes}^{iL/\hbar}\right) \text{ mod } \mathcal{J}_0. \quad (93)$$

Proof. Since $h(y) = 1$ on $\text{supp } f$, we have

$$\int dy h(y) (\mathcal{O}(x_i), \Phi^\dagger(y)) (\Phi(y), \mathcal{O}(x_j)) = \frac{1}{2} (\mathcal{O}(x_i), \mathcal{O}(x_j)), \quad x_i, x_j \in \text{supp } f. \quad (94)$$

Also, for $F_i = \int f \wedge \mathcal{O}_i$, it holds

$$\int dy h(y) (F_1, \Phi^\dagger(y)) (\Phi(y), F_2) = \frac{1}{2} (F_1, F_2). \quad (95)$$

We now look at the divergence of the classical BRST current which using (17) can be written

$$d\mathbf{J}(y) = (\hat{S}_0 + L, \Phi(y)) (\Phi^\dagger(y), \hat{S}_0 + L), \quad y \in M_T. \quad (96)$$

Expanding in powers of λ , this leads to

$$d\mathbf{J}_0(y) = (\hat{S}_0, \Phi(y)) (\Phi^\dagger(y), \hat{S}_0) \quad y \in M_T. \quad (97)$$

$$d\mathbf{J}_{\text{int}}(y) = (\hat{S}_0, \Phi(y)) (\Phi^\dagger(y), L) + (\Phi \leftrightarrow \Phi^\dagger) + (L, \Phi(y)) (\Phi^\dagger(y), L), \quad y \in M_T. \quad (98)$$

Since $T_1(d\mathbf{J}_0) = 0$, the free BRST charge $Q_0 = \int_M \gamma \wedge T_1(\mathbf{J}_0)$ is independent of the choice of $\gamma(y)$. We choose two different γ 's, namely γ_+ , and γ_- with $\gamma_+(y) - \gamma_-(y) = dh(y)$ and

$$\text{supp}(\gamma_+) \subset J^+(\{x_1, \dots, x_n\}), \quad \text{supp}(\gamma_-) \subset J^-(\{x_1, \dots, x_n\}). \quad (99)$$

Then, we can write modulo \mathcal{J}_0

$$\begin{aligned} & \int dy h(y) T_{n+1}(d\mathbf{J}_0(y) \otimes \mathbf{L}_1(x_1) \otimes \dots \otimes \mathbf{L}_1(x_n)) \\ &= - \int dy dh(y) T_{n+1}(\mathbf{J}_0(y) \otimes \mathbf{L}_1(x_1) \otimes \dots \otimes \mathbf{L}_1(x_n)) \\ &= - \int dy (\gamma_+(y) - \gamma_-(y)) T_{n+1}(\mathbf{J}_0(y) \otimes \mathbf{L}_1(x_1) \otimes \dots \otimes \mathbf{L}_1(x_n)) \\ &= -T_n(\mathbf{L}_1(x_1) \otimes \dots \otimes \mathbf{L}_1(x_n)) \star \int dy \gamma_+(y) \wedge T_1(\mathbf{J}_0(y)) \\ &\quad + \int dy \gamma_-(y) \wedge T_1(\mathbf{J}_0(y)) \star T_n(\mathbf{L}_1(x_1) \otimes \dots \otimes \mathbf{L}_1(x_n)) \\ &= T_1(Q_0) \star T_n(\mathbf{L}_1(x_1) \otimes \dots \otimes \mathbf{L}_1(x_n)) - T_n(\mathbf{L}_1(x_1) \otimes \dots \otimes \mathbf{L}_1(x_n)) \star T_1(Q_0) \\ &= [Q_0, T_n(\mathbf{L}_1(x_1) \otimes \dots \otimes \mathbf{L}_1(x_n))]_\star, \end{aligned}$$

where, we have made use of the causal factorization axiom (37). Multiplying by $\frac{i^n}{n! \hbar^n}$, integrating against $f(x_1) \dots f(x_n)$ and summing over n , we arrive at

$$\int dy h(y) T(d\mathbf{J}_0(y) \otimes e_{\otimes}^{iL/\hbar}) = [Q_0, T(e_{\otimes}^{iL/\hbar})]. \quad (100)$$

Since $(\hat{S}_0, \hat{S}_0) = 0$, in particular the relation (95) implies

$$\begin{aligned} \frac{1}{2}(\hat{S}_0 + L, \hat{S}_0 + L) &= (\hat{S}_0, L) + \frac{1}{2}(L, L) \\ &= \int dy h(y) (\hat{S}_0, \Phi(y)) (\Phi^\dagger(y), L) + (\Phi \leftrightarrow \Phi^\dagger) + (L, \Phi(y)) (\Phi^\dagger(y), L) \\ &= \int dy h(y) d\mathbf{J}_{\text{int}}(y). \end{aligned} \quad (101)$$

Adding (100) and $\int dy h(y) T(d\mathbf{J}_{\text{int}}(y) \otimes e_{\otimes}^{iL/\hbar}) = \int dy h(y) \frac{1}{2} T((\hat{S}_0 + L, \hat{S}_0 + L) \otimes e_{\otimes}^{iL/\hbar})$, we obtain (93). \square

Theorem 15 ([1]). *The interacting BRST current is conserved on-shell, i.e.*

$$d\mathbf{J}_I(x) = 0, \quad \forall x \in M, \text{ mod } \mathcal{J}_0. \quad (102)$$

Proof. According to the discussion of the algebraic adiabatic limit in section 3.2, it suffices to prove

$$T\left(d\mathbf{J}(y) \otimes e_{\otimes}^{iL/\hbar}\right) = 0, \quad y \in M_T, \quad (103)$$

for $L = \int f \mathbf{L}_{\text{int}}$, and where $M_T = (-T, T) \times \Sigma \subset \text{supp } f$ is the region where $f = 1$. This relation is to be understood as a formal power series. To prove it, we observe that (103) is violated at each order n by a term of the form $T_1(\alpha_n(y, x_1, \dots, x_n))$, where each $\alpha_n \in \mathbf{P}^{4/4/\dots/4}(M^{n+1})$.

Now, from proposition 14 and the Ward identity for the case $F = L$, it follows that if there exists a renormalization scheme in which $A(e_{\otimes}^L) = 0$, then

$$T\left(\int dy h(y) d\mathbf{J}(y) \otimes e_{\otimes}^{iL/\hbar}\right) = 0. \quad (104)$$

This means that $\int \alpha_n(y, x_1, \dots, x_n) dy = 0$ (since $h(y) = 1$ on M_T), and consequently

$$\alpha_n(y, x_1, \dots, x_n) = d_y \beta_n(y, x_1, \dots, x_n), \quad (105)$$

for some $\beta_n \in \mathbf{P}^{3/4/\dots/4}(M^{n+1})$. This β_n can then be used for changing to a new scheme by setting

$$D_{n+1}(\mathbf{J}_0 \otimes \mathbf{L}_1(x_1) \otimes \dots \otimes \mathbf{L}_1(x_n)) = \beta_n(y, x_1, \dots, x_n). \quad (106)$$

By similar arguments as in the proof of theorem 12, it then follows that in the new scheme the violation α_n is absent. \square

4 Quantum BRST charge and the algebra of interacting quantum fields

In the previous section, we have outlined the construction of $\hat{\mathbf{W}}_L$, the interacting algebra of observables corresponding to the enlarged classical theory defined by the action \hat{S} . Clearly, $\hat{\mathbf{W}}_L$ contains all interacting fields including non-gauge-invariant and unphysical ones. We now intend to define the algebra of physical, gauge invariant interacting quantum fields \mathcal{F}_I as a subalgebra of $\hat{\mathbf{W}}_L$. To this end, we first need to understand how $[\mathbf{Q}_L, -]$ acts on $\hat{\mathbf{W}}_L$ which is described in section 4.1. Second, we need to verify that $[\mathbf{Q}_L, -]$ acts as a nilpotent (graded) derivation on $\hat{\mathbf{W}}_L$ which is done in the next section 4.2

4.1 Action of $[\mathbf{Q}_L, -]$ on quantum fields

In this part, we prove our main theorem which yields a general expression for the commutator of the interacting quantum BRST charge and an interacting time-ordered product of n composite fields $T_{L,n}(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n) \in \hat{\mathbf{W}}_L$ defined in (54). Note, however, that the local S-matrix $T(e_{\otimes}^{iL/\hbar})$ is not of the form (54). Before proving our main theorem, we, thus, state the following which expresses that \mathbf{Q}_L commutes with $T(e_{\otimes}^{iL/\hbar})$.

Theorem 16. *If $A(e_{\otimes}^L) = 0$, then we have*

$$[\mathbf{Q}_L, T(e_{\otimes}^{iL/\hbar})] = 0, \quad \text{mod } \mathcal{J}_0. \quad (107)$$

Proof. We make use of the fact that since \mathbf{J}_L is conserved, $\mathbf{Q}_L = \int_M \gamma \wedge \mathbf{J}_L$ is independent of the closed 1-form γ . We can thus make use of γ_+ and γ_- , defined in (99), which are supported in the future/past of support of L , and write

$$\begin{aligned}
[\mathbf{Q}_L, T(e_{\otimes}^{iL/\hbar})] &= T(e_{\otimes}^{iL/\hbar})^{-1} \star T(e_{\otimes}^{iL/\hbar} \otimes Q) \star T(e_{\otimes}^{iL/\hbar}) - T(e_{\otimes}^{iL/\hbar} \otimes Q) \\
&= T(e_{\otimes}^{iL/\hbar})^{-1} \star T(e_{\otimes}^{iL/\hbar} \otimes \int \gamma_+(y) \mathbf{J}(y)) \star T(e_{\otimes}^{iL/\hbar}) \\
&\quad - T(e_{\otimes}^{iL/\hbar} \otimes \int \gamma_-(y) \mathbf{J}(y)) \\
&= T_1(\int \gamma_+(y) \mathbf{J}(y)) \star T(e_{\otimes}^{iL/\hbar}) - T(e_{\otimes}^{iL/\hbar} \otimes \int \gamma_-(y) \mathbf{J}(y)) \\
&= T(e_{\otimes}^{iL/\hbar} \otimes \int \gamma_+(y) \mathbf{J}(y)) - T(e_{\otimes}^{iL/\hbar} \otimes \int \gamma_-(y) \mathbf{J}(y)) \\
&= T(e_{\otimes}^{iL/\hbar} \otimes \int dh(y) \mathbf{J}(y)) \\
&= -T(e_{\otimes}^{iL/\hbar} \otimes \int h(y) d\mathbf{J}(y)) \\
&= 0.
\end{aligned}$$

In the third line we have used the causal factorization property of T_n , to factorize T_{n+1} into $T_1 \star T_n$ and $T_n \star T_1$, and in the forth line we have recombined T_1 and T_n again into T_{n+1} . Finally, we have made use of equation (104) in the last line. \square

In order to prove theorem 18, we need the following lemma which is a generalization of proposition 14 to the case with insertion of n local operators $\mathcal{O}_1, \dots, \mathcal{O}_n$.

Lemma 17. *If $A(e_{\otimes}^L) = 0$, then $d\mathbf{J}(y)$ integrated against $h(y)$ satisfies modulo \mathcal{J}_0*

$$\begin{aligned}
&T(\int h(y) d\mathbf{J}(y) \otimes \mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\
&= [Q_0, T(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar})] + \frac{1}{2} T((\hat{S}_0 + L, \hat{S}_0 + L) \otimes \mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}).
\end{aligned} \tag{108}$$

Proof. The proof is similar to the proof of proposition 14. We have,

$$\begin{aligned}
&T(\int h(y) \wedge d\mathbf{J}_0(y) \otimes \mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\
&= T(\int (\gamma_- - \gamma_+) \wedge \mathbf{J}_0(y) \otimes \mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\
&= -T(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \star T_1(\int \gamma_+ \wedge \mathbf{J}_0(y)) \\
&\quad + T_1(\int \gamma_- \wedge \mathbf{J}_0(y)) \star T(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar})
\end{aligned} \tag{109}$$

$$\begin{aligned}
&= T_1(Q_0) \star T(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) - T(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \star T_1(Q_0) \\
&= [Q_0, T(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar})].
\end{aligned} \tag{110}$$

Also using (101), we find

$$\begin{aligned}
&\frac{1}{2} T((\hat{S}_0 + L, \hat{S}_0 + L) \otimes \mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\
&= T(\int h(y) \wedge d\mathbf{J}_1(y) \otimes \mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}).
\end{aligned} \tag{111}$$

Adding (110) and (111), we obtain the claimed equation. \square

We are now in a position to prove our main theorem.

Theorem 18. *If $A(e_{\otimes}^L) = 0$, then we have modulo \mathcal{J}_0*

$$\begin{aligned} [\mathbf{Q}_L, T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)] &= [Q_0, T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)] \\ &\quad + R_{n,1}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n; \frac{1}{2}(\hat{S}_0 + L, \hat{S}_0 + L) \otimes e_{\otimes}^{iL/\hbar}), \end{aligned} \quad (112)$$

where Q_0 is the BRST charge of the free theory.

Proof. The proof follows essentially the same type of arguments which was used for the proof of the theorem 16. We have

$$\begin{aligned} &[\mathbf{Q}_L, T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)] \\ &= T(e_{\otimes}^{iL/\hbar})^{-1} \star T(Q \otimes e_{\otimes}^{iL/\hbar}) \star T(e_{\otimes}^{iL/\hbar})^{-1} \star T(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\ &\quad - T(e_{\otimes}^{iL/\hbar})^{-1} \star T(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \star T(e_{\otimes}^{iL/\hbar})^{-1} \star T(Q \otimes e_{\otimes}^{iL/\hbar}) \\ &= T(e_{\otimes}^{iL/\hbar})^{-1} \star T(e_{\otimes}^{iL/\hbar} \otimes \int \gamma_-(y) \mathbf{J}(y)) \star T(e_{\otimes}^{iL/\hbar})^{-1} \star T(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\ &\quad - T(e_{\otimes}^{iL/\hbar})^{-1} \star T(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \star T(e_{\otimes}^{iL/\hbar})^{-1} \star T(e_{\otimes}^{iL/\hbar} \otimes \int \gamma_+(y) \mathbf{J}(y)) \\ &= T(e_{\otimes}^{iL/\hbar})^{-1} \star T_1(\int \gamma_-(y) \mathbf{J}(y)) \star T(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\ &\quad - T(e_{\otimes}^{iL/\hbar})^{-1} \star T(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \star T_1(\int \gamma_+(y) \mathbf{J}(y)) \\ &= T(e_{\otimes}^{iL/\hbar})^{-1} \star T(\int \gamma_-(y) \mathbf{J}(y) \otimes \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\ &\quad - T(e_{\otimes}^{iL/\hbar})^{-1} \star T(\int \gamma_+(y) \mathbf{J}(y) \otimes \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\ &= -T(e_{\otimes}^{iL/\hbar})^{-1} \star T(\int dh(y) \mathbf{J}(y) \otimes \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\ &= T(e_{\otimes}^{iL/\hbar})^{-1} \star T(\int h(y) d\mathbf{J}(y) \otimes \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \\ &= T(e_{\otimes}^{iL/\hbar})^{-1} \star \left([Q_0, T(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar})] \right. \\ &\quad \left. + T(\frac{1}{2}(\hat{S}_0 + L, \hat{S}_0 + L) \otimes \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iL/\hbar}) \right) \\ &= [Q_0, T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)] - \frac{1}{2}(\hat{S}_0 + L, \hat{S}_0 + L)_L \star T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n) \\ &\quad + T_{L,n+1}(\frac{1}{2}(\hat{S}_0 + L, \hat{S}_0 + L) \otimes \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n), \end{aligned}$$

where we have used (108). Now upon using (58), we arrive at (112). \square

Theorem 18 gives an expression for the commutator of interacting BRST charge in terms of the free BRST charge and an extra term. We would now like to derive an explicit formula for the commutator of \mathbf{Q}_L and quantum fields without any reference to Q_0 , and express it in terms of the quantum BRST differential (3), quantum anti-bracket (4) and anomalies with higher than 2 insertions (72). This is given in the following corollary.

Corollary 19. *If $A(e_{\otimes}^L) = 0$, then we have modulo \mathcal{J}_0*

$$\begin{aligned} \frac{1}{i\hbar}[\mathcal{Q}_L, T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)] &= \sum_{i=1}^n T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \hat{\mathcal{S}}\mathcal{O}_i \otimes \cdots \otimes \mathcal{O}_n) \\ &\quad + \frac{\hbar}{i} \sum_{I_2}^n T_{L,n-1}((\mathcal{O}_i, \mathcal{O}_j)_{i,j \in I_2} \otimes \bigotimes_{k \in I_2^c} \mathcal{O}_k) \\ &\quad + \sum_I \left(\frac{\hbar}{i}\right)^{|I|-1} T_{L,n-|I|+1} \left(\hat{A}_{|I|} \left(\bigotimes_{i \in I} \mathcal{O}_i \right) \otimes \bigotimes_{j \in I^c} \mathcal{O}_j \right), \end{aligned} \quad (113)$$

where the sum runs over all non-empty partitions I of the set $\{1, 2, \dots, n\}$ and I^c is the complement partition, and $|I_2| = 2$. The above identity reduces to (1) and (2) for $n = 1$ and $n = 2$, respectively.

Proof. The proof follows from the anomalous Ward identity (66), and Theorem 18. We first prove (113) for the cases $n = 1$ and $n = 2$ in full details, and then outline the proof for $n > 3$. To prove (1), we calculate

$$\begin{aligned} \hat{s}_0 T(\mathcal{O} \otimes e_{\otimes}^{iF/\hbar}) &= \frac{\hbar}{i} \frac{d}{d\tau} \hat{s}_0 T(e_{\otimes}^{i(F+\tau\mathcal{O})/\hbar}) \Big|_{\tau=0} \\ &= \frac{\hbar}{i} \frac{d}{d\tau} \left(\frac{i}{2\hbar} T((\hat{S}_0 + F + \tau\mathcal{O}, \hat{S}_0 + F + \tau\mathcal{O}) \otimes e_{\otimes}^{i(F+\tau\mathcal{O})/\hbar}) \right) \Big|_{\tau=0} \\ &\quad + \frac{\hbar}{i} \frac{d}{d\tau} \left(\frac{i}{\hbar} T(A(e_{\otimes}^{F+\tau\mathcal{O}}) \otimes e_{\otimes}^{i(F+\tau\mathcal{O})/\hbar}) \right) \Big|_{\tau=0} \\ &= T((\hat{S}_0 + F, \mathcal{O}) + A(e_{\otimes}^F \otimes \mathcal{O})) \otimes e_{\otimes}^{iF/\hbar} \\ &\quad + \frac{i}{\hbar} T(\mathcal{O} \otimes \left(\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + A(e_{\otimes}^F) \right) \otimes e_{\otimes}^{iF/\hbar}). \end{aligned} \quad (114)$$

We, therefore, have

$$\begin{aligned} \hat{s}_0 \mathcal{O}_L &= \hat{s}_0 T(e_{\otimes}^{iL/\hbar})^{-1} \star T(\mathcal{O} \otimes e_{\otimes}^{iL/\hbar}) + T(e_{\otimes}^{iL/\hbar})^{-1} \star \hat{s}_0 T(\mathcal{O} \otimes e_{\otimes}^{iL/\hbar}) \\ &= -\frac{i}{\hbar} T(e_{\otimes}^{iL/\hbar})^{-1} \star \hat{s}_0 T(e_{\otimes}^{iL/\hbar}) \star \mathcal{O}_L + T(e_{\otimes}^{iL/\hbar})^{-1} \star \hat{s}_0 T(\mathcal{O} \otimes e_{\otimes}^{iL/\hbar}) \\ &= \left((\hat{S}_0 + L, \mathcal{O}) + A(e_{\otimes}^L \otimes \mathcal{O}) \right)_L + \frac{i}{\hbar} R(\mathcal{O}; \frac{1}{2}(\hat{S}_0 + L, \hat{S}_0 + L) \otimes e_{\otimes}^{iL/\hbar}), \end{aligned} \quad (115)$$

where we have used the anomalous Ward identity (66), and equation (114) for the case $F = L$, and $A(e_{\otimes}^L) = 0$. Going on-shell, that is replacing \hat{s}_0 with $\frac{1}{i\hbar}[Q_0, -]$, we obtain modulo \mathcal{J}_0

$$[Q_0, \mathcal{O}_L] = i\hbar \left((\hat{S}_0 + L, \mathcal{O}) + A(e_{\otimes}^L \otimes \mathcal{O}) \right)_L - R(\mathcal{O}; \frac{1}{2}(\hat{S}_0 + L, \hat{S}_0 + L) \otimes e_{\otimes}^{iL/\hbar}). \quad (116)$$

Now upon using (112) we arrive at

$$[\mathcal{Q}_L, \mathcal{O}_L(x)] = i\hbar \left((\hat{S}_0 + L, \mathcal{O}(x)) + A(e_{\otimes}^L \otimes \mathcal{O}(x)) \right)_L, \quad \text{mod } \mathcal{J}_0. \quad (117)$$

However, according to the remark 10, since both $(S_0 + L, \mathcal{O}(x))$ and $A(e_{\otimes}^L \otimes \mathcal{O}(x))$ are localized at x , we can safely set $f = 1$ in L in both of them and thus get $(S_0 + L, \mathcal{O}(x)) +$

$A(e_{\otimes}^L \otimes \mathcal{O}(x)) = \hat{q}\mathcal{O}(x)$ which leads to (1).

To prove (2), we first derive the following relation which is obtained in an analogous way by differentiating the anomalous Ward identity twice:

$$\begin{aligned}
\hat{s}_0 T(\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes e_{\otimes}^{iF/\hbar}) &= \frac{\hbar}{i} \frac{d}{d\tau} \hat{s}_0 T(\mathcal{O}_1 \otimes e_{\otimes}^{i(F+\tau\mathcal{O}_2)/\hbar}) \Big|_{\tau=0} \\
&= T\left((((\hat{S}_0 + F, \mathcal{O}_1) + A(e_{\otimes}^F \otimes \mathcal{O}_1)) \otimes \mathcal{O}_2 \right. \\
&\quad \left. + \mathcal{O}_1 \otimes ((\hat{S}_0 + F, \mathcal{O}_2) + A(e_{\otimes}^F \otimes \mathcal{O}_2))) \otimes e_{\otimes}^{iF/\hbar} \right) \\
&\quad + \frac{\hbar}{i} T(((\mathcal{O}_1, \mathcal{O}_2) + A(\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes e_{\otimes}^F)) \otimes e_{\otimes}^{iF/\hbar}) \\
&\quad + \frac{i}{\hbar} T(\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes (\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + A(e_{\otimes}^F)) \otimes e_{\otimes}^{iF/\hbar}),
\end{aligned} \tag{118}$$

which in turn leads to

$$\begin{aligned}
\hat{s}_0 T_{L,2}(\mathcal{O}_1 \otimes \mathcal{O}_2) &= T_{L,2}(\hat{q}\mathcal{O}_1 \otimes \mathcal{O}_2 + \mathcal{O}_1 \otimes \hat{q}\mathcal{O}_2) + \frac{\hbar}{i} ((\mathcal{O}_1, \mathcal{O}_2)_{\hbar})_L \\
&\quad + \frac{i}{\hbar} R(\mathcal{O}_1 \otimes \mathcal{O}_2; \frac{1}{2}(\hat{S}_0 + L, \hat{S}_0 + L) \otimes e_{\otimes}^{iL/\hbar}).
\end{aligned} \tag{119}$$

Now, substituting the above relation in (112) and safely setting $f = 1$, and replacing \hat{s}_0 with $\frac{1}{i\hbar}[Q_0, -]$, we arrive at (2).

Similarly, the proof for $n \geq 3$, follows from

$$\begin{aligned}
&\hat{s}_0 T(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iF/\hbar}) \\
&= \sum_{i=1}^n T(\mathcal{O}_1 \otimes \cdots \otimes (\hat{S}_0 + F, \mathcal{O}_i) \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iF/\hbar}) \\
&\quad + \frac{\hbar}{i} \sum_{I_2}^n T_{L,n-1}((\mathcal{O}_i, \mathcal{O}_j)_{i,j \in I_2} \otimes \bigotimes_{k \in I_2^c} \mathcal{O}_k \otimes e_{\otimes}^{iF/\hbar}) \\
&\quad + \sum_I \left(\frac{\hbar}{i}\right)^{|I|-1} T_{n-|I|+1} \left(A_{|I|} \left(\bigotimes_{i \in I} \mathcal{O}_i \otimes e_{\otimes}^{iF/\hbar} \right) \otimes \bigotimes_{j \in I^c} \mathcal{O}_j \otimes e_{\otimes}^{iF/\hbar} \right), \\
&\quad - T\left(\left(\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + A(e_{\otimes}^F) \right) \otimes \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \otimes e_{\otimes}^{iF/\hbar} \right) = 0.
\end{aligned} \tag{120}$$

$$\tag{121}$$

We prove the above equation inductively. For $n = 3$, by replacing F with $F + \tau_i \mathcal{O}_1 + \tau_2 \mathcal{O}_2 + \tau_3 \mathcal{O}_3$ in (66), differentiating with respect to $\tau_3 \tau_2 \tau_1$, and setting $\tau_1 = \tau_2 = \tau_3 = 0$,

we find

$$\begin{aligned}
\hat{s}_0 T(\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3 \otimes e_{\otimes}^{iF/\hbar}) = & T\left(\left((\hat{S}_0 + F, \mathcal{O}_1) + A(e_{\otimes}^F \otimes \mathcal{O}_1)\right) \otimes \mathcal{O}_2 \otimes \mathcal{O}_3\right. \\
& + \mathcal{O}_1 \otimes \left((\hat{S}_0 + F, \mathcal{O}_2) + A(e_{\otimes}^F \otimes \mathcal{O}_2)\right) \otimes \mathcal{O}_3 \\
& + \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \left((\hat{S}_0 + F, \mathcal{O}_3) + A(e_{\otimes}^F \otimes \mathcal{O}_3)\right) \otimes e_{\otimes}^{iF/\hbar} \\
& + \frac{\hbar}{i} T(\mathcal{O}_1 \otimes (A(e_{\otimes}^F \otimes \mathcal{O}_2 \otimes \mathcal{O}_3) + (\mathcal{O}_2, \mathcal{O}_3)) \otimes e_{\otimes}^{iF/\hbar}) \\
& + \frac{\hbar}{i} T(\mathcal{O}_2 \otimes (A(e_{\otimes}^F \otimes \mathcal{O}_1 \otimes \mathcal{O}_3) + (\mathcal{O}_1, \mathcal{O}_3)) \otimes e_{\otimes}^{iF/\hbar}) \\
& + \frac{\hbar}{i} T(\mathcal{O}_3 \otimes (A(e_{\otimes}^F \otimes \mathcal{O}_1 \otimes \mathcal{O}_2) + (\mathcal{O}_1, \mathcal{O}_2)) \otimes e_{\otimes}^{iF/\hbar}) \\
& + \left(\frac{\hbar}{i}\right)^2 T(A(e_{\otimes}^F \otimes \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3) \otimes e_{\otimes}^{iF/\hbar}) \\
& - T\left(\left(\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + A(e_{\otimes}^F)\right)\right. \\
& \left. \otimes \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3 \otimes e_{\otimes}^{iF/\hbar}\right), \tag{122}
\end{aligned}$$

which is precisely (121) for $n = 3$. Now assume that (121) holds at order $l > 3$. Replacing everywhere F with $F + \tau \mathcal{O}_{l+1}$, differentiating with respect to τ , setting $\tau = 0$ and using the relations (76), (77), (78) in this expression at order l , we find (121) at order $l + 1$, by straightforward but lengthy calculations similar to the proof of lemma 11, which we omit here. \square

Remark 20.

(1) Using the definition of \hat{q} and $(-, -)_{\hbar}$, we may equivalently write (113) in the following form

$$\begin{aligned}
\frac{1}{i\hbar} [\mathcal{Q}_L, T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)] = & \sum_{i=1}^n T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \hat{q}\mathcal{O}_i \otimes \cdots \otimes \mathcal{O}_n) \\
& + \frac{\hbar}{i} \sum_{1 \leq i < j}^n T_{L,n-1}(\mathcal{O}_1 \otimes \cdots \otimes (\mathcal{O}_i, \mathcal{O}_j)_{\hbar} \otimes \cdots \otimes \mathcal{O}_n) \\
& + \sum_{I_0 \cup \cdots \cup I_r \subset \underline{n}} T_{r+1}\left(\left(\frac{\hbar}{i}\right)^{|I_0|-1} \hat{A}_{|I_0|}(\otimes_{j \in I_0} \mathcal{O}_j) \otimes \bigotimes_{i \in I_k} \mathcal{O}_i\right), \tag{123}
\end{aligned}$$

where the sum runs over all partitions $I_0 \cup \cdots \cup I_r$ of the set $\underline{n} = \{1, \dots, n\}$ into pairwise disjoint non-empty subsets, with $|I_0| > 2$.

(2) Comparing (113) and (71), it readily follows that we can write the identity (113) also in terms of the generating functional of interacting time ordered products

$$T_L(e_{\otimes}^{iF/\hbar}) = \sum_{n=0}^{\infty} \frac{i^n}{\hbar^n n!} T_{L,n}(F^{\otimes n}), \tag{124}$$

where $T_{L,n}(F^{\otimes n})$ are defined in equation (54). With this, we have

$$[\mathcal{Q}_L, T_L(e_{\otimes}^{iF/\hbar})] = \frac{1}{2} T_L\left((\hat{S}_0 + F, \hat{S}_0 + F) \otimes e_{\otimes}^{iF/\hbar}\right) + T_L(\hat{A}(e_{\otimes}^F) \otimes e_{\otimes}^{iF/\hbar}), \tag{125}$$

which leads to identity (113) for all n . Note that (125) is similar to the anomalous Ward identity (66) only with free BRST differential $i\hbar\hat{s}_0$, (free) time-ordered products T and anomaly A being replaced with $[Q_L, -]$, interacting time-ordered products T_L and \hat{A} respectively. For this reason, we call the identity (125) the interacting anomalous Ward identity.

4.2 Nilpotency of $[Q_L, -]$

We have so far derived how the quantum field, Q_L , associated with the classical Noether charge Q of the BRST symmetry acts on arbitrary quantum fields via the \star -commutator. It necessarily has to give the correct classical limit as \hbar goes to zero. Indeed, since $A(e_{\otimes}^I \otimes \mathcal{O}(x))$ is of order $O(\hbar)$, from the expression (1) it follows that

$$\lim_{\hbar \rightarrow 0} \frac{i}{\hbar} [Q_L, -] = \hat{s}. \quad (126)$$

However, giving the correct classical limit is not a sufficient condition for $[Q_L, -]$ to define the action of the BRST symmetry on interacting quantum fields; in addition, it has to be nilpotent. To prove its nilpotency, we first need the following corollary of lemma 11.

Corollary 21. *Let $A(e_{\otimes}^L) = 0$. Then, the following identities hold.*

$$\hat{q}^2 = 0, \quad (127)$$

$$\hat{q}(\mathcal{O}_1, \mathcal{O}_2)_{\hbar} = (\hat{q}\mathcal{O}_1, \mathcal{O}_2)_{\hbar} - (\mathcal{O}_1, \hat{q}\mathcal{O}_2)_{\hbar}, \quad (5)$$

$$\begin{aligned} & ((\mathcal{O}_1, \mathcal{O}_2)_{\hbar}, \mathcal{O}_3)_{\hbar} + ((\mathcal{O}_2, \mathcal{O}_3)_{\hbar}, \mathcal{O}_1)_{\hbar} + ((\mathcal{O}_3, \mathcal{O}_1)_{\hbar}, \mathcal{O}_2)_{\hbar} + \hat{q}\hat{A}_3(\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3) \\ & + \hat{A}_3(\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \hat{q}\mathcal{O}_3) + \hat{A}_3(\hat{q}\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3) + \hat{A}_3(\mathcal{O}_1 \otimes \hat{q}\mathcal{O}_2 \otimes \mathcal{O}_3). \end{aligned} \quad (128)$$

Proof. To prove (127), we use the consistency condition (73) for anomalies which in the case of $n = 1$ gives

$$\hat{q}\hat{A}_1(\mathcal{O}) + \hat{A}_1(\hat{s}\mathcal{O}) = 0. \quad (129)$$

and calculate

$$\begin{aligned} \hat{q}^2\mathcal{O} &= \hat{q}\hat{s}\mathcal{O} + \hat{q}\hat{A}_1(\mathcal{O}) \\ &= \hat{s}^2\mathcal{O} + \hat{A}_1(\hat{s}\mathcal{O}) - \hat{A}_1(\hat{s}\mathcal{O}) \\ &= 0. \end{aligned}$$

For the proof of the identity (5) we need (73) for $n = 2$ which reads

$$\begin{aligned} & \hat{s}\hat{A}_2(\mathcal{O}_1 \otimes \mathcal{O}_2) + (\mathcal{O}_1, \hat{A}(\mathcal{O}_2)) - (\hat{A}(\mathcal{O}_1), \mathcal{O}_2) \\ & + \hat{A}_2(\hat{q}\mathcal{O}_1 \otimes \mathcal{O}_2) + \hat{A}_2(\mathcal{O}_1 \otimes \hat{q}\mathcal{O}_2) + \hat{A}_1((\mathcal{O}_1, \mathcal{O}_2)_{\hbar}) = 0. \end{aligned} \quad (130)$$

where the minus sign in the third term appears because of the symmetry property of the classical anti-bracket (equation (160)) and the fact that $\hat{A}(\mathcal{O}_1)$ is fermionic. Now adding $\hat{s}(\mathcal{O}_1, \mathcal{O}_2) - (\hat{s}\mathcal{O}_1, \mathcal{O}_2) + (\mathcal{O}_1, \hat{s}\mathcal{O}_2) = 0$ (see equation (162)) to the above equation, and noting that since $\hat{q}\mathcal{O}_1$ is fermionic $(\hat{q}\mathcal{O}_1, \mathcal{O}_2)_{\hbar} = (\hat{q}\mathcal{O}_1, \mathcal{O}_2) - \hat{A}_2(\hat{q}\mathcal{O}_1 \otimes \mathcal{O}_2)$ by definition (176), we arrive at (5). To prove (128), consider (73) for the particular case of $n = 3$:

$$\begin{aligned} & \hat{q}\hat{A}_3(\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3) \\ & + \hat{A}_3(\hat{q}\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3) + \hat{A}_2(\mathcal{O}_1 \otimes (\mathcal{O}_2, \mathcal{O}_3)_{\hbar}) + (\mathcal{O}_1, \hat{A}_2(\mathcal{O}_2 \otimes \mathcal{O}_3))_{\hbar} + \text{perm.} = 0. \end{aligned} \quad (131)$$

Adding the Jacobi identity, $(\mathcal{O}_1, (\mathcal{O}_2, \mathcal{O}_3)) + \text{perm.} = 0$, to the above equation we arrive at (128). \square

Note that the identity (128) implies that, contrary to the classical anti-bracket $(-, -)$, the quantum anti-bracket does not satisfy the Jacobi identity.

Using (127), it is now easy to see that $[\mathbf{Q}_L, -]$, when acting on \mathcal{O}_L is nilpotent,

$$[\mathbf{Q}_L, [\mathbf{Q}_L, \mathcal{O}_L]] = (\hat{q}^2 \mathcal{O})_L = 0. \quad (132)$$

One can then verify that it is also nilpotent when acts on the product of n interacting fields, since

$$[\mathbf{Q}_L, \mathcal{O}_{1L} \star \cdots \star \mathcal{O}_{nL}] = \sum_i \mathcal{O}_{1L} \star \cdots \star (\hat{q} \mathcal{O}_i)_L \star \cdots \star \mathcal{O}_{nL}, \quad (133)$$

and applying once again $[\mathbf{Q}_L, -]$ on both sides, using $\hat{q}^2 = 0$, and the fact that $[\mathbf{Q}_L, -]$ is a graded derivation (see equation (168)) and $\hat{q} \mathcal{O}_i$ are Grassmann odd, we find

$$[\mathbf{Q}_L, [\mathbf{Q}_L, \mathcal{O}_{1L} \star \cdots \star \mathcal{O}_{nL}]] = 0. \quad (134)$$

It thus remains to show that $[\mathbf{Q}_L, -]$ acting on $T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)$, is also nilpotent.

Theorem 22. *If $A(e_\otimes^L) = 0$, then $[\mathbf{Q}_L, [\mathbf{Q}_L, T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)]] = 0$, for all $\mathcal{O}_i \in \mathbf{P}(M)$.*

Proof. We have already proved the statement for $n = 1$. Let us before proving the claim for all n , explicitly verify it for $n = 2$. Using $\hat{q}^2 = 0$ and (5), we have

$$\begin{aligned} \frac{1}{(i\hbar)^2} [\mathbf{Q}_L, [\mathbf{Q}_L, T_{L,2}(\mathcal{O}_1 \otimes \mathcal{O}_2)]] &= T_{L,2}(\hat{q} \mathcal{O}_1 \otimes \hat{q} \mathcal{O}_2 - \hat{q} \mathcal{O}_1 \otimes \hat{q} \mathcal{O}_2) \\ &\quad + \frac{\hbar}{i} ((\mathcal{O}_1, \hat{q} \mathcal{O}_2)_\hbar - (\hat{q} \mathcal{O}_1, \mathcal{O}_2)_\hbar + \hat{q}(\mathcal{O}_1, \mathcal{O}_2)_\hbar)_L \\ &= 0, \end{aligned} \quad (135)$$

where in the second line the minus sign appears because of the identity (175) and the fact that $\hat{q} \mathcal{O}_1$ is fermionic, and the last equality follows from the identity (5). We prove the claim for all n , by applying $[\mathbf{Q}_L, -]$ on both sides of the interacting anomalous Ward identity (125), which is a generating functional for the identity (113) for all n , and we obtain

$$\begin{aligned} &[\mathbf{Q}_L, [\mathbf{Q}_L, T_L(e_\otimes^{iF/\hbar})]] \\ &= [\mathbf{Q}_L, T_L((\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + \hat{A}(e_\otimes^F)) \otimes e_\otimes^{iF/\hbar})] \\ &= \frac{\hbar}{i} \frac{d}{d\tau} [\mathbf{Q}_L, T_L(e_\otimes^{i(F + \tau(\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + \hat{A}(e_\otimes^F))) / \hbar})] \Big|_{\tau=0} \\ &= T_L(((S_0 + F, (\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + \hat{A}(e_\otimes^F))) e_\otimes^{iF/\hbar}) \\ &\quad + T_L(\hat{A}((\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + \hat{A}(e_\otimes^F)) \otimes e_\otimes^F) \otimes e_\otimes^{iF/\hbar}) \\ &\quad + T_L((\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + \hat{A}(e_\otimes^F)) \otimes (\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + \hat{A}(e_\otimes^F)) e_\otimes^{iF/\hbar}) \\ &= T_L((\hat{S}_0 + F, \hat{A}(e_\otimes^F)) + \hat{A}((\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + \hat{A}(e_\otimes^F)) \otimes e_\otimes^F) \otimes e_\otimes^{iF/\hbar}) \\ &= 0, \end{aligned}$$

where in the forth line, $(\hat{S}_0 + F, (\hat{S}_0 + F, \hat{S}_0 + F)) = 0$ by Jacobi identity, the sixth line vanishes using the graded symmetry of $T_{L,n}$ and the fact that $\frac{1}{2}(\hat{S}_0 + F, \hat{S}_0 + F) + \hat{A}(e_\otimes^F)$ is fermionic, and the last line vanishes by the consistency condition (80) for \hat{A} . \square

Hilbert space representation

We have now collected all the tools which are required to represent the algebra of observables on a Hilbert space \mathcal{H}_I . While the on-shell algebra $\hat{\mathcal{F}}_I$, defined in (62), can be represented on a space with an indefinite inner product, here we define the physical algebra of observables which can indeed be represented on a Hilbert space.

It is shown in [24], [1] that \mathcal{H}_I can be constructed via a deformation process. For this construction to work, a crucial requirement is that the quantum BRST charge squares to zero, i.e. $\mathbf{Q}_L^2 = 0$. We prove this as a corollary of the nilpotency of $[\mathbf{Q}_L, -]$.

Corollary 23. *If $A(e_\otimes^L) = 0$, the quantum BRST charge is nilpotent, i.e. $\mathbf{Q}_L^2 = 0$.*

Proof. Since \mathbf{Q}_L is odd, $[\mathbf{Q}_L, \mathbf{Q}_L] = 2\mathbf{Q}_L \star \mathbf{Q}_L \equiv 2\mathbf{Q}_L^2$. From the nilpotency of $[\mathbf{Q}_L, -]$ and the graded Jacobi identity, we have

$$0 = [\mathbf{Q}_L, [\mathbf{Q}_L, \mathcal{O}_L]] = \frac{1}{2} [\mathcal{O}_L, [\mathbf{Q}_L, \mathbf{Q}_L]] = [\mathbf{Q}_L^2, \mathcal{O}_L] \quad \text{for all } \mathcal{O}_L \in \hat{\mathbf{W}}_L.$$

By the proposition 2.1 in [8], it follows that \mathbf{Q}_L^2 must be a multiple of the identity element $\mathbf{1}$ in $\hat{\mathbf{W}}_L$

$$\mathbf{Q}_L^2 = k\mathbf{1}, \tag{136}$$

for some constant scalar k with dimension 0, and ghost number 2 made out of background fields. However, there is no such a constant in the theory, thus $k = 0$. \square

Finally, upon taking the algebraic adiabatic limit which defines $[\mathbf{Q}_I, -]$ at the end of calculations, we define the on-shell, physical algebra of gauge invariant observables by

$$\mathcal{F}_I(M, g) := \frac{\ker[\mathbf{Q}_I, -] \cap \hat{\mathcal{F}}_I}{\text{im}[\mathbf{Q}_I, -] \cap \hat{\mathcal{F}}_I} \quad \text{at } q = 0, \tag{137}$$

where $\hat{\mathcal{F}}_I$ is defined in (62). Now, using $\mathbf{Q}_L^2 = 0$, it can be shown that $\mathcal{F}_I(M, g)$ can be represented on a Hilbert space \mathcal{H}_I

$$\pi_I : \mathcal{F}_I \rightarrow \text{End}(\mathcal{H}_I), \tag{138}$$

where all the anti-fields are represented with the zero element in $\text{End}(\mathcal{H}_I)$

$$\pi_I(\Phi^\dagger) = 0. \tag{139}$$

Remark 24.

(1) In [1], the existence of the algebra \mathcal{F}_I as the cohomology of $[\mathbf{Q}_I, -]$ and the fact that it admits a Hilbert space representation is proven in the opposite way of our proof; it is first shown that $\mathbf{Q}_L^2 = 0$, from which it follows that $[\mathbf{Q}_L, -]$ is nilpotent. However in that reference, $\mathbf{Q}_L^2 = 0$ follows from the Ward identity (65) for the case $F = L + \int f \wedge \mathbf{J}$ which is potentially violated by the anomaly $A(\mathbf{J} \otimes e_\otimes^L) \in H_2^3(\hat{s}|d, M)$. Based on similar arguments discussed in remark 13, this anomaly can be shown to be the zero element in this cohomology class for the case of pure Yang-Mills theory, but is not necessarily the case in many other theories with local gauge symmetry. On the contrary in our analysis, the necessary condition for $\mathbf{Q}_L^2 = 0$ is only the triviality of $H_1^4(\hat{s}|dM)$.

(2) Enquiring the form of $[Q_0, T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)]$ which is derived in the proof of theorem 19, one finds out that these expressions for all n contain (1) a part made out of \hat{q} , $(-, -)_\hbar$ and \hat{A}_n , where all the fields in the arguments of $T_{L,n}$ are supported on the diagonal and hence one can safely set the cutoff $f = 1$ without encountering any IR problem, and (2) a second part which is $-\frac{1}{2}R_{n,1}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n; (\hat{S}_0 + L, \hat{S}_0 + L) \otimes e_\otimes^{iL/\hbar})$. The terms in the arguments of this second expression, however, is not supported on the diagonal and hence setting $f = 1$ is not well defined. Nevertheless, our main formula (112) is essentially stating that the $[Q_L, -]$ differs from $[Q_0, -]$ exactly by $+\frac{1}{2}R_{n,1}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n; (\hat{S}_0 + L, \hat{S}_0 + L) \otimes e_\otimes^{iL/\hbar})$, and hence in the expressions in corollary 19 such terms are absent, i.e. those expressions are IR-finite.

(3) If one, nevertheless, formally sets $f = 1$ in L , in the second term in the right hand side of (112), then $(\hat{S}_0 + L, \hat{S}_0 + L) = (S, S) = 0$, and formally $\frac{1}{2}R_{n,1}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n; (\hat{S}_0 + L, \hat{S}_0 + L) \otimes e_\otimes^{iL/\hbar}) \rightarrow 0$. Therefore

$$[Q_L, T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)] = [Q_0, T_{L,n}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n)]_\star, \quad (\text{FORMALLY}), \quad (140)$$

that is the commutator of the interacting BRST charge and interacting fields coincides with the commutator of the free BRST charge with them.

(4) A similar formal conclusion follows from Ward identity (65) for the invariance of the local S -matrix under free BRST charge

$$[Q_0, T(e_\otimes^{iL/\hbar})] = 0, \quad (\text{FORMALLY}). \quad (141)$$

However, as we have shown in theorem 107, the local S -matrix commutes with the interacting BRST charge without encountering any IR problem:

$$[Q_L, T(e_\otimes^{iL/\hbar})] = 0. \quad (142)$$

4.3 Quantum gauge invariant observables

As discussed in the previous section, at the classical level, one can recover the gauge invariant observables as elements of the \hat{s} -cohomology class at ghost number zero. We would now like to gain insight about the nature of gauge invariant quantum observables, i.e. those in the quotient algebra \mathcal{F}_I .

To understand the nature of elements in \mathcal{F}_I , we observe that it follows obviously from (1) that if $\hat{q}\mathcal{O} = 0$, then $[Q_L, \mathcal{O}_L] = 0$. However, as can be seen in the proof of proposition 25, $[Q_L, \mathcal{O}_L] = 0$ does not necessarily imply $\hat{q}\mathcal{O} = 0$. This is due to the fact that $[Q_L, \mathcal{O}_L] = 0$ is an identity for renormalized quantities which are defined only up to a certain ambiguity, as opposed to $\hat{q}\mathcal{O} = 0$ which is a classical, unambiguous identity. One can, therefore, take advantage of the renormalization ambiguity in its definition and may possibly set $(A(e_\otimes^L \otimes \mathcal{O}))_L$ to zero, although $A(e_\otimes^L \otimes \mathcal{O}) \neq 0$.

In general, the cohomology classes of \hat{q} are smaller than the cohomology classes of \hat{s} . This follows from fact that \hat{q} can be seen to form a filtration starting with \hat{s} . A filtration is a way of decomposing operators and correspondingly the nilpotent differential acting on the space of such operators. For example, decomposition of the classical operators and the classical BRST differential in powers of the coupling constant, $\mathcal{O} = \mathcal{O}_0 + \lambda\mathcal{O}_1 + \dots$,

$\hat{s} = \hat{s}_0 + \lambda \hat{s}_1 + \dots$ is a particular way of filtration. Then, by well-known theorems in homological perturbation theory (see e.g. [25]), it follows that the cohomology of the full differential (here \hat{s}) is a subset of the cohomology of the lowest order part (here \hat{s}_0). Now, in the case of quantum BRST differential we naturally have a filtration of the space $\mathbf{P}(M)[[\hbar]]$ into powers of \hbar , and

$$\hat{q} = \hat{s} + \hbar \hat{q}_1 + \hbar^2 \hat{q}_2 + \dots \quad (143)$$

Therefore, elements in the cohomology of \hat{q} , are also in the cohomology of \hat{s} .

We would now like to ask the opposite questions: given a classical observable in the kernel of \hat{s} , is the corresponding interacting quantum field in the kernel of $[\mathbf{Q}_L, -]$? In the following proposition, we see that this is indeed the case.

Proposition 25. *Let \mathcal{O} be a classical gauge-invariant operator, i.e. $\hat{s}\mathcal{O}(x) = 0$, of form degree p and ghost number 0. If the cohomology ring $H_1^p(\hat{s})$ is trivial, then $[\mathbf{Q}_L, \mathcal{O}(x)_L] = 0$.*

Proof. Since $\hat{s}\mathcal{O} = 0$, we have $[\mathbf{Q}_L, \mathcal{O}_L] = (A(e_{\otimes}^L \otimes \mathcal{O}))_L$. We proceed by showing that the “obstruction”, $(A(e_{\otimes}^L \otimes \mathcal{O}(x)))_L$, for $[\mathbf{Q}_L, \mathcal{O}(x)_L]$ to vanish, can be removed by passing to a new renormalization scheme. Note that contrary to the proof of the Ward identity (65) for the case of $F = L$, where we first had to show that $A(e_{\otimes}^L) = 0$ in a renormalization scheme which in turn implies $A(e_{\otimes}^L) = 0$, here we only have to show that $A(e_{\otimes}^L \otimes \mathcal{O}(x)) = 0$ for $F = L$ since there is an insertion of the local operator $\mathcal{O}(x)$ (see remark 10 part 2).

Due to nilpotency of $[\mathbf{Q}_L, -]$, we can derive a consistency condition for $A(e_{\otimes}^L \otimes \mathcal{O}(x))$, namely

$$0 = [\mathbf{Q}_L, [\mathbf{Q}_L, \mathcal{O}(x)_L]] = [\mathbf{Q}_L, (A(e_{\otimes}^L \otimes \mathcal{O}(x)))_L]. \quad (144)$$

Let $A^m(\mathcal{O}(x) \otimes e_{\otimes}^L)$ be the first non-trivial term in the \hbar expansion of $A(e_{\otimes}^L \otimes \mathcal{O}(x))$. The above equation to lowest order in \hbar , gives

$$\hat{s}A^m(\mathcal{O}(x) \otimes e_{\otimes}^L) = 0. \quad (145)$$

Since \mathcal{O} has ghost number 0, $A^m(x)$ has ghost number 1 and thus it belongs to $H_1^p(\hat{s}, M)$ which is empty. Therefore,

$$A^m(\mathcal{O}(x) \otimes e_{\otimes}^L) = \hat{s}b^m(x), \quad (146)$$

for some $b^m(x) \in H_0^p(\hat{s}, M)$. We then expand $b^m = \sum_n \frac{\lambda^n}{n!} b_n^m$ and use b_n^m to make a redefinition of the renormalization scheme using the finite local counter terms

$$D_n(\mathcal{O}(x_1) \otimes \mathbf{L}_1(x_2) \otimes \dots \otimes \mathbf{L}_1(x_n)) = -\hbar^m b_n^m(x_1) \delta(x_2, \dots, x_n). \quad (147)$$

In the new scheme \tilde{T} , it turns out (see [1]) that the new and old anomalies are related by

$$\begin{aligned} \tilde{A}^m(\mathcal{O}(x) \otimes e_{\otimes}^L) &= A^m(\mathcal{O}(x) \otimes e_{\otimes}^L) + \hat{s}D^m(\mathcal{O}(x) \otimes e_{\otimes}^L) \\ &= a^m(x) - \hat{s}b^m(x) \\ &= 0. \end{aligned}$$

Therefore, in the new scheme the anomaly vanishes to lowest order in \hbar . Iterating the argument, one can show that the anomaly vanishes to all orders in \hbar . \square

Remark 26. For the case of the pure Yang-Mills theory, when G is semi-simple with no abelian factors, $H(\hat{s}, M)$ is generated by elements of the form [16]

$$\prod_k r_{t_k}(g, R, \nabla R, \dots, \nabla^k R) \prod_i p_{r_i}(C) \prod_j \Theta_{r_j}(F, \mathcal{D}F, \dots, \mathcal{D}^l F) \quad (148)$$

where p_r and Θ_s are invariant polynomials of the Lie-algebra of G and r_t is a local functional of the metric g , the Riemann tensor R and its derivatives. However, at ghost number 1 the above expression vanishes as there is no invariant monomial p_r , and thus $H_1^p(\hat{s})$ is trivial. Therefore by the above proposition 25, $[Q_L, \mathcal{O}_L] = 0$, for $\mathcal{O} \in H_0^p(\hat{s})$.

5 Comparison with other approaches

5.1 Batalin-Vilkovisky formalism in the path integral approach

There are obvious differences between our approach and the Path integral approach. In the latter, one defines the generating functional for the correlation functions via an integral over the infinite dimensional manifold of all field configurations. Such a path integral is a priori only formal in many respects: the measure on the infinite dimensional space does not exist, and if one wants to make sense of it as a formal power series in the coupling constant each individual term suffers from both IR and UV divergences. Even ignoring these difficulties, the path integral is a state-dependent quantity as it generates correlation functions in a specific state. This makes it difficult to appreciate the local and covariant nature of the renormalization ambiguities [9] in arbitrary curved space-times with no preferred state. Despite those differences, there are also formal similarities between the two approaches in studying gauge theories. They both lead to a kind of algebraic structure which is called the Batalin-Vilkovisky (BV) algebra. Let us briefly outline the types of arguments in the path integral formalism which result in the emergence of the BV-structure, and then compare that with our formalism.

In the path integral approach to gauge theories (see e.g. [25], [26]), one modifies the “measure” by higher order terms in \hbar to obtain a gauge invariant measure $\mathcal{D}\phi$. This corrections can be equivalently seen as quantum correction to the classical action S and classical observables \mathcal{O} , i.e.

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\phi \mathcal{O}_\hbar e^{iW/\hbar}, \quad (149)$$

where $\mathcal{O}_\hbar = \mathcal{O} + \hbar \mathcal{O}_1 + \hbar^2 \mathcal{O}_2 + \dots$ and where $W = S + \hbar W_1 + \hbar^2 W_2 + \dots$ is called the *quantum action*. The precise form of W is determined as a solution to the quantum master equation (QME):

$$(W, W) = 2i\hbar \Delta W, \quad (150)$$

which ensures that $\langle \mathcal{O} \rangle$ is independent of the gauge fixing fermion ψ . In the above equation, $\Delta = \frac{\delta}{\delta\phi\delta\phi^\dagger}$ is the so-called *BV laplacian*. Using Δ , one defines the *quantum BRST differential*

$$\sigma = (W, -) + i\hbar \Delta \quad (151)$$

which satisfies the following properties:

- (i) $\sigma^2 = 0$,

$$(ii) \quad \sigma(\mathcal{O}_1 \mathcal{O}_2) = (\sigma \mathcal{O}_1) \mathcal{O}_2 + \mathcal{O}_1 (\sigma \mathcal{O}_2) + (\mathcal{O}_1, \mathcal{O}_2),$$

$$(iii) \quad \sigma(\mathcal{O}_1, \mathcal{O}_2) = (\sigma \mathcal{O}_1, \mathcal{O}_2) + (\mathcal{O}_1, \sigma \mathcal{O}_2).$$

Once the quantum Master equation is satisfied, one can prove the following *Ward identity for correlation functions*

$$\sum_{i=1}^n \langle \mathcal{O}_1 \dots \sigma \mathcal{O}_i \dots \mathcal{O}_n \rangle = 0, \quad (152)$$

and for $n = 1$ is reduced to $\langle \sigma \mathcal{O} \rangle = 0$.

We now point out the following analogies and differences between the path integral approach and ours.

(1) The definitions of our quantum BRST differentials \hat{q} defined in equation (3) and σ given in equation (151) are similar in that they both are given by classical BRST \hat{s} plus higher \hbar corrections. However, they are different in that the quantum corrections for σ is given by $\hbar((W_1, -) + i\Delta) + \sum_{n \geq 2} \hbar^n (W_n, -)$ which are ill-defined since Δ is a singular operator and W_n are in general IR divergent, while the quantum corrections for \hat{q} are given by $i\hbar \hat{A}(-)$ defined in equation (72) which is well-defined. Indeed, as first noted by authors of [27] (however, see section 5.2 below), $\hat{A}(-)$ may be seen as the “renormalized BV laplacian”.

(2) The properties (i), (ii), (iii) of σ define the BV algebra. Evidently, (i) and (iii) are similar to properties (127) and (5) of \hat{q} , and property (ii) is similar to (2). The difference between them is the presence of quantum anti-bracket $(-, -)_\hbar$ which differs from the classical anti-bracket by terms of order $O(\hbar^2)$ (which are precisely given by (4)). Therefore, one may see our BV data (127), (5), and (2) as defining the “renormalized BV algebra”.

(3) The QME is in general violated by potential anomalies, and as it turns out in the path integral approach, such anomalies belong to the same cohomological class as $A(e_\otimes^L)$ (see equation (82)). Therefore, our proof that $A(e_\otimes^L) = 0$, and hence the Ward identity (65) holds, may be taken as the counterpart for the proof that the QME holds in the path integral framework.

(4) As we have proved in the present work, from our Ward identity (65) it ultimately follows that $[\mathbf{Q}_L, -]$ is a nilpotent derivation and hence one can define the algebra of physical observables as the cohomology of $[\mathbf{Q}_L, -]$. Of course observables in the image of \hat{q} are quotiented out and their expectation value in a physical state $|\Psi\rangle \in \mathcal{H}_{\text{phys}}$ vanishes: $\langle \hat{q}\mathcal{O} \rangle_\Psi = 0$. This fact is clearly comparable with equation (152) for the case of one operator insertion which states that the expectation value of observables in the image of σ vanishes if the quantum action satisfies the QME. For n operator insertions, we obtain from (123)

$$\sum_{i=1}^n T \langle \mathcal{O}_1 \dots \hat{q} \mathcal{O}_i \dots \mathcal{O}_n \rangle_\Psi \quad (153)$$

$$+ \sum_{1 \leq i < j}^n T \langle \mathcal{O}_1 \dots (\mathcal{O}_i, \mathcal{O}_j)_\hbar \dots \mathcal{O}_n \rangle_\Psi + \sum_{I_0 \cup \dots \cup I_r \subset \underline{n}} T \langle \hat{A}_{|I_0|}(\otimes_{j \in I_0} \mathcal{O}_j) \prod_{i \in I_k} \mathcal{O}_i \rangle_\Psi = 0, \quad (154)$$

where $T\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_\Psi := \langle \Psi | T_{I,n}(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n) | \Psi \rangle$ are renormalized time-ordered n point functions of the theory in the state $|\Psi\rangle \in \mathcal{H}_{\text{phys}}$. Comparison with (152) reveals that the identity (154) involves quantum corrections to (152) and may be interpreted as the *renormalized Ward identity for correlation functions*.

5.2 Batalin-Vilkovisky formalism in the pAQFT approach

The closest approach to ours is the BV formalism in the framework of perturbative algebraic quantum field theory (pAQFT) developed in [27]. While this approach is in the same spirit as ours, there are still notable differences which we point out here.

(1) In the pAQFT approach, contrary to (21), one makes a different split of the action into free \tilde{S}_0 and interacting \tilde{S}_1 parts by putting all terms depending on the anti-fields into \tilde{S}_1 . Although this does not affect the classical BRST differential, i.e. $\hat{s} = (\tilde{S}_0 + \tilde{S}_1, -) = (S_0 + S_1, -)$, the free action \tilde{S}_0 only acts on anti-fields, i.e. $(\tilde{S}_0, \Phi^\dagger(x)) = \frac{\delta \tilde{S}_0}{\delta \Phi(x)}$ and $(\tilde{S}_0, \Phi) = 0$. One then formulates a similar anomalous Ward identity in the form

$$(\tilde{S}_0, T(e_\otimes^{iF/\hbar})) = \frac{i}{\hbar} \left(T\left(\frac{1}{2}(\tilde{S}_0 + F, \tilde{S}_0 + F) \otimes e_\otimes^{iF/\hbar}\right) - i\hbar T(\Delta(F) \otimes e_\otimes^{iF/\hbar}) \right), \quad (155)$$

where $\Delta(F)$ is the anomaly. Despite the obvious similarity to (66), a key difference is that, the left hand side of the above identity vanishes on on-shell field configurations, contrary to (66). To clarify the issue let us elaborate on the proof of our anomalous Ward identity given in [1]. One first decomposes $\hat{s} = s + \sigma$ where s is the BRST differential which only acts on fields, and σ is the Koszul-Tate differential which acts only on anti-fields. The anomalous Ward identity is then obtained by adding two different identities: (1) an identity for $s_0 T(e_\otimes^{iF/\hbar})$ which gives an anomaly term $\delta(e_\otimes^F)$ and (2) an identity, originally derived in [28], for $\sigma_0 T(e_\otimes^{iF/\hbar})$ which gives an anomaly term $\Delta(e_\otimes^F)$. One then defines $A(e_\otimes^F) = \delta(e_\otimes^F) + \Delta(e_\otimes^F)$ and obtains (66). It seems that the identity (155) is the second identity mentioned above which realizes the free Koszul-Tate differential on the local S-matrix.

(2) The quantum BV operator \tilde{s} in the pAQFT approach is defined by

$$\tilde{s} := R_V^{-1} \circ (\tilde{S}_0, -) \circ R_V, \quad (156)$$

where $R_V = T(e_\otimes^{iV/\hbar})^{-1} \star T(e_\otimes^{iV/\hbar} \otimes -)$ is viewed here as an operator (the quantum Möller operator) which takes a functional \mathcal{O} and gives \mathcal{O}_V . This definition differs from \hat{q} in two respects: (1) From the nilpotency of $(\tilde{S}_0, -)$, it follows that $\tilde{s}^2 = 0$, which means that \tilde{s} is always nilpotent by construction, irrespective of the presence or absence of an anomaly. Evidently, this is different from our quantum BRST differential \hat{q} which is nilpotent only if $A(e_\otimes^L) = 0$. (2) Since $(\tilde{S}_0, \mathcal{O})$ vanishes on on-shell configurations, equation (156) defines \tilde{s} essentially as an off-shell operator. This is again different from our definition of \hat{q} which is non-vanishing on both on-shell and off-shell field configurations. Nevertheless, using the QME, one can show that \tilde{s} takes the following form:

$$\tilde{s}\mathcal{O} := \hat{s}\mathcal{O} - \Delta_V(\mathcal{O}), \quad (157)$$

which is analogous to our definition of \hat{q} , except for the difference between $\Delta_V(\mathcal{O})$ and $\hat{A}_1(\mathcal{O})$ which was explained in point 1.

(3) In [29], it is shown that the quantum BV operator, on-shell, can be written as the commutator with an interacting charge Q , i.e.

$$[R_V(Q), R_V(\mathcal{O})] = i\hbar R_V(\tilde{s}\mathcal{O}). \quad (158)$$

As pointed out above, on-shell, where the above formula is valid, the right hand side vanishes. Therefore, the formula (158) seems to express that the charge $R_V(Q)$ commutes with *all* interacting fields on-shell. This is of course a true statement for Q being the generator of the Koszul-Tate differential. However, this is obviously different from identity (1) which expresses that the interacting BRST charge \mathbf{Q}_L only commutes with interacting fields \mathcal{O}_L for which \mathcal{O} is in the kernel of \hat{q} . Consequently, being in the cohomology of $[R_V(Q), -]$ does not seem to provide a criterion for selecting the physical observables and for selecting the physical states of the theory in a Hilbert space representation.

6 Outlook

In this paper, we have developed new algebraic structures in quantum gauge theories which enables one to construct the algebra of renormalized gauge-invariant observables in a model-independent fashion. Such structures, namely quantum BRST differential $\hat{q} = \hat{s} + O(\hbar)$ and quantum anti-bracket $(-, -)_\hbar = (-, -) + O(\hbar)$ are indeed analogous to the classical ones modified with certain quantum corrections. The new structures seem to provide sufficient tools to investigate further open issues in gauge theories. Here, we outline two of those issues which can be the subject of future research

Gauge-fixing independence

In section 2, we pointed out that for perturbative quantization of gauge theories one has to choose a particular way to fix the gauge in order to render the equations of motion hyperbolic. The natural question is, then, whether and in which sense different quantum field theories defined with different (in general, non-linear) gauge-fixings are equivalent?

To be more specific, different gauge-fixings may arise, for instance, from a family gauge-fixing fermions $\psi(\xi)$, for $\xi \in (0, 1]$ with

$$\psi(\xi) = \int_M \bar{C}_I (\nabla^\mu A_\mu^I + \frac{\xi}{2} B^I), \quad (159)$$

which gives rise to the family of linear covariant gauges. $\xi = 1$ corresponds to the Feynman gauge (which was considered in this work) and the limit $\xi \rightarrow 0$ corresponds to the Landau gauge. In this case, the question of the equivalence of quantum field theories defined with $\psi(\xi)$ and $\psi(\xi')$ may be stated as follows. At the classical level, there exists an isomorphism $\mathcal{O} \mapsto e^{(-, \delta\psi)} \mathcal{O}$, with $\delta\psi = \psi(\xi') - \psi(\xi)$ between the cohomologies of the BRST differentials \hat{s}_ξ and $\hat{s}_{\xi'}$ which ensures that the observables of the two theories are in one-to-one correspondence. Based on the analogy between classical and quantum structures worked out in the paper, one can then formulate [30] the gauge-fixing independence at the quantum level as the existence of an isomorphism $\mathcal{O}_{L(\xi)} \mapsto e^{(-, \delta\psi)_\hbar} \mathcal{O}_{L(\xi')}$ between \hat{q}_ξ and $\hat{q}_{\xi'}$ cohomologies.

Background independence

As discussed in section 2, a classical gauge theory is the dynamical theory of a gauge-connection \mathcal{D} which is subject to Yang-Mills equations. In perturbative quantum field

theory, however, one splits the gauge connection $\mathcal{D} = \nabla + i\lambda A$ into a background connection ∇ and a perturbation A which is a Lie algebra-valued one form, and quantizes the latter in perturbation theory. The natural question is then does the whole construction of QFT depend on the particular form of this split?

In fact, the same question can be posed in the case of a self-interacting scalar field theory, where one might split the dynamical field $\Phi = \varphi + \phi$ into a background φ and a perturbation ϕ . In that case, it turns out [31] that the quantum theory is background independent in the sense that the variation with respect to φ (which are implemented via the action of a flat connection) of the φ -dependent interacting fields \mathcal{O}_L^φ (which are viewed as sections of an algebra-bundle over the manifold of background configurations) vanishes.

For the case of gauge theories, the tools developed in the body of the paper become crucial in analysing this issue. The strategy [32] is to calculate, in a mathematically precise sense, variations of the physical observables (i.e. those in the cohomology of quantum BRST differential) with respect to the background. In this case, the connection which implements the background variations, has to be (1) well-defined on the $[Q_L, -]$ -cohomology, and (2) flat, when acts on observables, only up to an element in the image of the quantum BRST differential. Moreover, in this case, one requires such a variation of observables to vanishes up to an element in the image of the quantum BRST differential which is not in the cohomology and hence is un-physical. If this turns out to be the case, the quantum gauge theory would be background independent.

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A Graded symmetries and derivations

Let ϵ_i be the Grassmann parity of the operator \mathcal{O}_i . This means $\epsilon_i = 0 \bmod 2$ if \mathcal{O}_i is bosonic (even), and $\epsilon_i = 1 \bmod 2$ if \mathcal{O}_i is fermionic (odd). First, note that \hat{s} and \hat{q} change the Grassmann parity of the operators they act upon. Let us begin with the classical anti-bracket. It has the following symmetry property

$$(\mathcal{O}_1, \mathcal{O}_2) = (-1)^{(\epsilon_1+1)(\epsilon_2+1)+1} (\mathcal{O}_2, \mathcal{O}_1). \quad (160)$$

and satisfies the graded Jacobi identity

$$\begin{aligned} (-1)^{(\epsilon_1+1)(\epsilon_3+1)} (\mathcal{O}_1, (\mathcal{O}_2, \mathcal{O}_3)) &+ (-1)^{(\epsilon_2+1)(\epsilon_1+1)} (\mathcal{O}_2, (\mathcal{O}_3, \mathcal{O}_1)) \\ &+ (-1)^{(\epsilon_3+1)(\epsilon_2+1)} (\mathcal{O}_3, (\mathcal{O}_1, \mathcal{O}_2)) = 0. \end{aligned} \quad (161)$$

From (160) and (161), together with $(S, \mathcal{O}) = \hat{s}\mathcal{O}$, it follows that

$$\hat{s}(\mathcal{O}_1, \mathcal{O}_2) = (\hat{s}\mathcal{O}_1, \mathcal{O}_2) - (-1)^{\epsilon_1}(\mathcal{O}_1, \hat{s}\mathcal{O}_2). \quad (162)$$

The graded Peierls bracket satisfies

$$\{\mathcal{O}_1, \mathcal{O}_2\} = (-1)^{\epsilon_1\epsilon_2}\{\mathcal{O}_2, \mathcal{O}_1\}, \quad (163)$$

and

$$\hat{s}\{\mathcal{O}_1, \mathcal{O}_2\} = \{\hat{s}\mathcal{O}_1, \mathcal{O}_2\} + (-1)^{\epsilon_1}\{\mathcal{O}_1, \hat{s}\mathcal{O}_2\}. \quad (164)$$

The \star -commutator is defined to be

$$[\mathcal{O}_1, \mathcal{O}_2] = \mathcal{O}_1 \star \mathcal{O}_2 - (-1)^{\epsilon_1\epsilon_2}\mathcal{O}_2 \star \mathcal{O}_1, \quad (165)$$

which is graded

$$[\mathcal{O}_1, \mathcal{O}_2] = (-1)^{\epsilon_1\epsilon_2}[\mathcal{O}_2, \mathcal{O}_1], \quad (166)$$

and satisfies the graded Jacobi identity

$$(-1)^{\epsilon_1\epsilon_3}[\mathcal{O}_1, [\mathcal{O}_2, \mathcal{O}_3]] + (-1)^{\epsilon_3\epsilon_2}[\mathcal{O}_3, [\mathcal{O}_1, \mathcal{O}_2]] + (-1)^{\epsilon_1\epsilon_2}[\mathcal{O}_2, [\mathcal{O}_3, \mathcal{O}_1]] = 0. \quad (167)$$

From this identity, it then follows that

$$[\mathcal{Q}_L, \mathcal{O}_{1L} \star \cdots \star \mathcal{O}_{nL}] = \sum_k (-1)^{\sum_{l < k} \epsilon_l} \mathcal{O}_{1L} \star \cdots \star (\hat{q}\mathcal{O}_i)_L \star \cdots \star \mathcal{O}_{nL}. \quad (168)$$

We would now like to derive the identities (1), (2) and (113) when \mathcal{O}_i are either bosonic or fermionic. The starting point for that is the anomalous Ward identity. However, the anomalous Ward identity, as stated in theorem 9, only applies to bosonic F 's, as for fermionic ones $(\hat{S}_0 + F, \hat{S}_0 + F) = (\hat{S}_0, F) + (F, F) + (F, \hat{S}_0) = 0$. It turns out that this identity for local functionals F_i with Grassmann parity ϵ_i takes the form

$$\begin{aligned} \hat{s}_0 T_n(F_1 \otimes \cdots \otimes F_n) &= \sum_{k=0} (-1)^{\sum_{l < k} \epsilon_l} T_n(F_1 \otimes \cdots \otimes \hat{s}_0 F_k \otimes \cdots \otimes F_n) \\ &+ \frac{\hbar}{i} \sum_{I_2} (-1)^{\epsilon_{I_2} + \epsilon_i} T_{n-1}((F_i, F_j)_{i,j \in I_2} \otimes \bigotimes_{k \in I_2^c} F_k) \\ &+ \sum_I \left(\frac{\hbar}{i}\right)^{|I|-1} (-1)^{\epsilon_I} T_{n-|I|+1}(A_{|I|}(\bigotimes_{i \in I} F_i) \otimes \bigotimes_{j \in I^c} F_j), \end{aligned} \quad (169)$$

where I is a non-empty and ordered partition of the set $\{1, 2, \dots, n\}$ and I^c is the complement partition and $|I_2| = 2$, and where ϵ_{I_2} and ϵ_I are signs that are obtained by reordering F_i 's into $F_1 \dots F_n$, i.e.

$$\prod_{j \in I_2} F_j \prod_{k \in I_2^c} F_k = (-1)^{\epsilon_{I_2}} F_1 \dots F_n, \quad (170)$$

$$\prod_{j \in I} F_j \prod_{k \in I^c} F_k = (-1)^{\epsilon_I} F_1 \dots F_n. \quad (171)$$

The above identity is derived by (1) starting from (71) for bosonic $g_1 F_1, \dots, g_n F_n$, where g_i are anti-commuting numbers, $g_i F_j = (-1)^{\epsilon_i \epsilon_j} F_j g_i$, with the same Grassmann parity as F_i , and (2) using the following identities

$$(g_1 \mathcal{O}_1, g_2 \mathcal{O}_2) = (-1)^{(\epsilon_1+1)\epsilon_2} g_1 g_2 (\mathcal{O}_1, \mathcal{O}_2), \quad (172)$$

$$A_n(g_1 \mathcal{O}_1 \otimes \dots \otimes g_n \mathcal{O}_n) = (-1)^{\sum_i \epsilon_i + \sum_{i < j} \epsilon_i \epsilon_j} g_1 \dots g_n A_n(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n), \quad (173)$$

which are consequences of the symmetry property (160) and the graded symmetry of A_n .

The correct signs in the identities (1), (2) and (113) are now obtained along the same lines as the proof of corollary 19, but now starting from (169). It turns out that

$$\begin{aligned} \frac{1}{i\hbar} [\mathbf{Q}_L, T_{L,n}(\mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n)] &= \sum_{k=0} (-1)^{\sum_{l < k} \epsilon_l} T_{L,n}(\mathcal{O}_1 \otimes \dots \otimes \hat{s}\mathcal{O}_i \otimes \dots \otimes \mathcal{O}_n) \\ &\quad + \frac{\hbar}{i} \sum_{I_2} (-1)^{\epsilon_{I_2} + \epsilon_i} T_{L,n-1}((\mathcal{O}_i, \mathcal{O}_j)_{i,j \in I_2} \otimes \bigotimes_{k \in I_2^c} \mathcal{O}_k) \\ &\quad + \sum_I \left(\frac{\hbar}{i}\right)^{|I|-1} (-1)^{\epsilon_I} T_{L,n-|I|+1} \left(\hat{A}_{|I|} \left(\bigotimes_{i \in I} \mathcal{O}_i \right) \otimes \bigotimes_{j \in I^c} \mathcal{O}_j \right), \end{aligned} \quad (174)$$

where $I, I^c, I_2, I_2^c, \epsilon_{I_2}, \epsilon_I$ are defined below equation (169). For the case of $n = 1$ it gives (1) and for $n = 2$ it leads to

$$\frac{1}{i\hbar} [\mathbf{Q}_L, T_{L,2}(\mathcal{O}_1 \otimes \mathcal{O}_2)] = T_{L,2}(\hat{q}\mathcal{O}_1 \otimes \mathcal{O}_2 + (-1)^{\epsilon_1} \mathcal{O}_1 \otimes \hat{q}\mathcal{O}_2) + (-1)^{\epsilon_1} \frac{\hbar}{i} ((\mathcal{O}_1, \mathcal{O}_2)_\hbar)_L, \quad (175)$$

where the quantum anti-bracket is now defined to be

$$(\mathcal{O}_1, \mathcal{O}_2)_\hbar := (\mathcal{O}_1, \mathcal{O}_2) + (-1)^{\epsilon_1} \hat{A}_2(\mathcal{O}_1 \otimes \mathcal{O}_2). \quad (176)$$

It satisfies the same symmetry property as the classical anti-bracket

$$(\mathcal{O}_1, \mathcal{O}_2)_\hbar = (-1)^{(\epsilon_1+1)(\epsilon_2+1)+1} (\mathcal{O}_2, \mathcal{O}_1)_\hbar, \quad (177)$$

together with

$$\hat{q}(\mathcal{O}_1, \mathcal{O}_2)_\hbar = (\hat{q}\mathcal{O}_1, \mathcal{O}_2)_\hbar - (-1)^{\epsilon_1} (\mathcal{O}_1, \hat{q}\mathcal{O}_2)_\hbar, \quad (178)$$

and

$$\begin{aligned} &\hat{q}\hat{A}_3(\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3) + (-1)^{(\epsilon_1+1)(\epsilon_3+1)} \left((\mathcal{O}_1, (\mathcal{O}_2, \mathcal{O}_3)_\hbar)_\hbar + \hat{A}_3(\hat{q}\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3) \right) \\ &\quad + (-1)^{(\epsilon_2+1)(\epsilon_1+1)} \left((\mathcal{O}_2, (\mathcal{O}_3, \mathcal{O}_1)_\hbar)_\hbar + \hat{A}_3(\mathcal{O}_1 \otimes \hat{q}\mathcal{O}_2 \otimes \mathcal{O}_3) \right) \\ &\quad + (-1)^{(\epsilon_3+1)(\epsilon_2+1)} \left((\mathcal{O}_3, (\mathcal{O}_1, \mathcal{O}_2)_\hbar)_\hbar + \hat{A}_3(\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \hat{q}\mathcal{O}_3) \right) \\ &= 0. \end{aligned} \quad (179)$$

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